# Some exercises on coherent lower previsions

SIPTA school, September 2010

1. Consider the lower prevision given by:

| | $f(1)$ | $f(2)$ | $f(3)$ | $\underline{P}(f)$ |
|---|---|---|---|---|
| $f_1$ | 2 | 1 | 0 | 0.5 |
| $f_2$ | 0 | 1 | 2 | 1 |
| $f_3$ | 0 | 1 | 0 | 1 |

   (a) Does it avoid sure loss?

   *Answer.* Yes. It suffices to see that there is a linear prevision that dominates $\underline{P}$ on its domain. The prevision $P$ given by $P(f) = f(2)$ satisfies this. □

   (b) Is it coherent?

   *Answer.* No. Since $\underline{P}(f_3) = 1 = \max\{f_3(1), f_3(2), f_3(3)\}$, the only linear prevision that dominates $\underline{P}$ is precisely $P(f) = f(2)$ for all $f$. But $P$ does not coincide with $\underline{P}$ on all gambles: $P(f_1) = 1 > 0.5 = \underline{P}(f_1)$. Since $\underline{P}$ is not the lower envelope of the set $\mathcal{M}(\underline{P})$, we deduce that it is not coherent. □

2. Let $\underline{P}$ be the lower prevision on $\mathcal{L}(\{1, 2, 3\})$ given by

$$\underline{P}(f) = \frac{\min\{f(1), f(2), f(3)\}}{2} + \frac{\max\{f(1), f(2), f(3)\}}{2}.$$

   Is it coherent?

   *Answer.* No. Since $\underline{P}$ is defined on a linear space, a necessary condition for coherence is that it is supper-additive, meaning that $\underline{P}(f+g) \geq \underline{P}(f)+\underline{P}(g)$ for any pair of gambles $f, g$. To see that this does not hold, consider $f, g$ given by $f(1) = -1, f(2) = -1, f(3) = 1$ and $g(1) = -1, g(2) = 1, g(3) = -1$. Then $\underline{P}(f) = \underline{P}(g) = 0$, while $\underline{P}(f + g) = -1$. □

3. **Vacuous lower previsions.** Let $A$ be a non-empty subset of a (not necessarily finite) set $\mathcal{X}$. Say we only know that the lower probability of $A$ is equal to 1. This assessment is embodied through the lower prevision $\underline{P}$ defined on the singleton $\{I_A\}$ by $\underline{P}(A) = 1$ (again, recall that we denote $\underline{P}(I_A)$ by $\underline{P}(A)$).

(a) Show that the vacuous lower prevision relative to $A$, defined by

$$\underline{P}_A(f) := \inf_{x\in A} f(x)$$

for any $f \in \mathcal{L}(\mathcal{X})$, is a coherent lower prevision on $\mathcal{L}(\mathcal{X})$.

*Answer.* The domain of $\underline{P}_A$ is a linear space, so it suffices to check that

(i) $\underline{P}_A(f) \geq \inf_{x\in\mathcal{X}} f(x)$ for any $f \in \mathcal{L}(\mathcal{X})$,
(ii) $\underline{P}_A(\lambda f) = \lambda\underline{P}_A(f)$ for any $f \in \mathcal{L}(\mathcal{X})$ and any $\lambda > 0$, and
(iii) $\underline{P}_A(f+g) \geq \underline{P}_A(f) + \underline{P}_A(g)$ for any $f$, $g \in \mathcal{L}(\mathcal{X})$.

□

(b) Prove that the natural extension $\underline{E}$ of $\underline{P}$ is equal to the vacuous lower prevision relative to $A$:

$$\underline{E}(f) = \underline{P}_A(f) = \inf_{x\in A} f(x),$$

for any $f \in \mathcal{L}(\mathcal{X})$.

*Answer 1: Primal Approach.* $\underline{E}(f)$ is equal to the supremum achieved by the free variable $\gamma$ subject to the constraint

$$f - \gamma \geq \lambda(I_A - 1) \tag{0.1}$$

with variable $\lambda \geq 0$. Note that $\gamma = \inf_{x\in A} f(x)$ and $\lambda = \inf_{x\in A} f(x) - \inf_{x\in\mathcal{X}} f(x)$ yields a feasible solution of Eq. (0.1). Therefore, $\underline{E}(f) \geq \inf_{x\in A} f(x)$.
Let $\gamma$ and $\lambda$ constitute any feasible solution of Eq. (0.1). Then, since $\inf_{x\in A}$ is monotone, we find in particular that

$$\inf_{x\in A} f(x) - \gamma \geq \inf_{x\in A} \lambda(I_A(x) - 1)$$

Note that the right hand side is zero. Hence, $\gamma \leq \inf_{x\in A} f(x)$. Therefore, also $\underline{E}(f) \leq \inf_{x\in A} f(x)$. But, we already proved that $\underline{E}(f) \geq \inf_{x\in A} f(x)$, and hence, $\underline{E}(f) = \inf_{x\in A} f(x)$. □

*Answer 2: Dual Approach.* Again consider the set $\mathcal{M}(\underline{P})$ of linear previsions on $\mathcal{L}(\mathcal{X})$ that dominate $\underline{P}$. Now, if we can show that

$$\operatorname{ext}\mathcal{M}(\underline{P}) = \{\underline{P}_x : x \in A\}. \tag{0.2}$$

then the claim is established, since in that case

$$\underline{E}(f) = \inf_{Q\in\mathcal{M}(\underline{P})} Q(f) = \inf_{Q\in\operatorname{ext}\mathcal{M}(\underline{P})} Q(f) = \inf_{x\in A} \underline{P}_x(f) = \inf_{x\in A} f(x).$$

We shall prove Eq. (0.2) in case that $\mathcal{X}$ is a finite set (it holds in general case, but the proof becomes a bit more complex).

Assume that $\mathcal{X}$ is a finite set. To see that Eq. (0.2) holds, let $Q \in \mathcal{M}(\underline{P})$. Observe that

$$Q(\{x\}) = 1 - Q(\mathcal{X} \setminus \{x\}) \leq 1 - Q(A) = 0$$

for any $x \notin A$, and hence

$$\sum_{x \in A} Q(\{x\}) = \sum_{x \in \mathcal{X}} Q(\{x\}) = 1.$$

This implies that any $Q \in \mathcal{M}(\underline{P})$ is a convex mixture of $\underline{P}_x$ for $x \in A$ (see exercise on linear previsions):

$$Q(f) = \sum_{x \in \mathcal{X}} Q(\{x\})f(x) = \sum_{x \in A} Q(\{x\})f(x) = \sum_{x \in A} Q(\{x\})\underline{P}_x(f).$$

Since all $\underline{P}_x$ are linearly independent, Eq. (0.2) follows. □

*Answer 3: Least Committal Extension.* The statement is established if we show that $\underline{P}_A$ is the point-wise smallest coherent lower prevision on $\mathcal{L}(\mathcal{X})$ which dominates $\underline{P}$ on $\{I_A\}$.

Suppose $\underline{Q}$ is another coherent lower prevision on $\mathcal{L}(\mathcal{X})$ which dominates $\underline{P}$ on $\{I_A\}$, that is, $\underline{Q}(A) = 1$. Let $f \in \mathcal{L}(\mathcal{X})$. It is easy to check that

$$f \geq \underline{P}_A(f) + [\underline{P}_A(f) - \underline{P}_{\mathcal{X}}(f)](I_A - 1),$$

where $\underline{P}_{\mathcal{X}}(f) = \inf_{x \in \mathcal{X}} f(x)$. Since $\underline{Q}$ is coherent, this implies that

$$\begin{aligned}
\underline{Q}(f) &\geq \underline{Q}\big(\underline{P}_A(f) + [\underline{P}_A(f) - \underline{P}_{\mathcal{X}}(f)](I_A - 1)\big) \\
&= \underline{P}_A(f) + [\underline{P}_A(f) - \underline{P}_{\mathcal{X}}(f)]\underline{Q}(I_A - 1)\big) \\
&= \underline{P}_A(f),
\end{aligned}$$

since $\underline{Q}(I_A - 1) = \underline{Q}(A) - 1 = 0$. This establishes the proof. □

(c) Extra exercise. Each one of the questions (b), (c) and (d) can be solved in three different ways, either using

(i) the primal form—combinations of desirable gambles,
(ii) the dual form—sets of probability measures, or
(iii) the properties of coherence and natural extension, invoking the result proven in the preparatory exercise (a).

Invoke each one of the methods (i), (ii) and (iii) to answer each one of the questions (b), (c) and (d). You may cheat when solving question (d) using method (ii): it is much easier if you assume that $\mathcal{X}$ is finite.

4. Consider an urn with 10 balls, of which 3 are red, and the other 7 are either blue or yellow.

   (a) Determine the set $\mathcal{M}$ of linear previsions that represent the possible compositions of the urn.

   *Answer.* The set of possible compositions of the urn is given by the following table:

| Red | Blue | Yellow |
|---|---|---|
| 3 | 0 | 7 |
| 3 | 1 | 6 |
| 3 | 2 | 5 |
| 3 | 3 | 4 |
| 3 | 4 | 3 |
| 3 | 5 | 2 |
| 3 | 6 | 1 |
| 3 | 7 | 0 |

   It produces the following sets of linear previsions (we give the probability mass functions which are their restrictions to events):

| $P_i$ | Red | Blue | Yellow |
|---|---|---|---|
| $P_1$ | 3 | 0 | 7 |
| $P_2$ | 3 | 1 | 6 |
| $P_3$ | 3 | 2 | 5 |
| $P_4$ | 3 | 3 | 4 |
| $P_5$ | 3 | 4 | 3 |
| $P_6$ | 3 | 5 | 2 |
| $P_7$ | 3 | 6 | 1 |
| $P_8$ | 3 | 7 | 0 |

   □

   (b) Let $f$ be a gamble given by $f(blue) = 2, f(red) = 1, f(yellow) = -1$. Which is the lower prevision of $f$?

   *Answer.* The lower prevision of $f$ is the lower envelope of the set $\{P_1(f), P_2(f), \dots, P_8(f)\}$, which in this case is equal to $\underline{P}(f) = 0.3 \cdot 1 - 0.7 \cdot 1 = 0.4$. □

   (c) Do the same for an arbitrary gamble $g$.

*Answer.* Again, we have $\underline{P}(f) = \min\{P_1(f), P_2(f), \ldots, P_8(f)\}$; since $P_2, \ldots, P_7$ are convex combinations of $P_1, P_8$, it follows that

$$\begin{aligned}\underline{P}(f) &= \min\{p_1(f), P_8(f)\} \\ &= \min\{0.3f(\mathit{blue}) + 0.7f(\mathit{yellow}), 0.3f(\mathit{blue}) + 0.7f(\mathit{red})\} \\ &= 0.3f(\mathit{blue}) + 0.7\min\{f(\mathit{yellow}), f(\mathit{red})\}.\end{aligned}$$

□

5. Let $\mathcal{X} = \{1, 2, 3\}$, and consider the following sets of desirable gambles:

$$\begin{aligned}\mathcal{K}_1 &:= \{f : f(1) + f(2) + f(3) \geq 0\} \\ \mathcal{K}_2 &:= \{f : \max\{f(1), f(2), f(3)\} \geq 0\}.\end{aligned}$$

(a) Are $\mathcal{K}_1, \mathcal{K}_2$ coherent?

*Answer.* To see that the set of gambles $\mathcal{K}_1$ is coherent, we check that if verifies axioms (D1)–(D5) (note that because we are dealing with finite spaces, we can use maximum instead of supremum and minimum instead of infimum):

(D1) Take a gamble $f$ such that $\max f < 0$; then $f(1) + f(2) + f(3) \leq 3\max f < 0$, whence $f \notin \mathcal{K}_1$.

(D2) Conversely, if $f \geq 0$ then $f(1) + f(2) + f(3) \geq 0$, whence $f \in \mathcal{K}_1$.

(D3) If $f, g \in \mathcal{K}_1$, then $f(1) + f(2) + f(3) \geq 0$ and $g(1) + g(2) + g(3) \geq 0$. As a consequence, $(f + g)(1) + (f + g)(2) + (f + g)(3) = f(1) + f(2) + f(3) + g(1) + g(2) + g(3) \geq 0$, and this means that $f + g \in \mathcal{K}_1$.

(D4) Consider $f \in \mathcal{K}_1$ and $\lambda > 0$; then since $f(1) + f(2) + f(3) \geq 0$, we deduce that $(\lambda f)(1) + (\lambda f)(2) + (\lambda f)(3) = \lambda(f(1) + f(2) + f(3)) \geq 0$. This implies that $\lambda f \in \mathcal{K}_1$.

(D5) Assume that $f + \epsilon$ belongs to $\mathcal{K}_1$ for every $\epsilon > 0$; then $(f + \epsilon)(1) + (f + \epsilon)(2) + (f + \epsilon)(3) = f(1) + f(2) + f(3) + 3\epsilon \geq 0$ for all $\epsilon > 0$, whence $f(1) + f(2) + f(3) \geq -3\epsilon\ \forall \epsilon > 0$ and therefore $f(1) + f(2) + f(3) \geq 0$. This means that $f \in \mathcal{K}_1$.

To see that the set $\mathcal{K}_2$ is not coherent, it suffices to note that it does not satisfy axiom (D3): consider the gambles $f, g$ given by

$$f(1) = 1, f(2) = f(3) - 2; \quad g(2) = 1, g(1) = g(3) = -2;$$

then it holds that $\max f = \max g = 1$, whence both $f$ and $g$ belong to $\mathcal{K}_2$; however, $(f + g)(1) = (f + g)(2) = -1, (f + g)(3) = -4$, whence $f + g \notin \mathcal{K}_2$.

□

(b) If they are, which is the lower prevision they induce on the gamble $f$ given by $f(1) = 2, f(2) = 3, f(3) = -1$?

*Answer.* We only need to verify it for $\mathcal{K}_1$. Taking into account the correspondence between sets of desirable gambles and coherent lower previsions, we have that

$$\begin{aligned}\underline{P}(f) &= \sup\{\mu : f - \mu \in \mathcal{K}_1\} \\ &= \sup\{\mu : (f-\mu)(1) + (f-\mu)(2) + (f-\mu)(3) \geq 0\} \\ &= \sup\{\mu : f(1) + f(2) + f(3) - 3\mu \geq 0\} \\ &= \sup\{\mu : 4 - 3\mu \geq 0\} = \frac{4}{3}.\end{aligned}$$

□

6. Consider $\mathcal{X} = \{1, 2, 3\}$ and let $\mathcal{D}$ be the gambles given by

$$\mathcal{D} := \{(1, -1, 0), (0, 1, -1), (1, 0, -1)\}.$$

   (a) Calculate the set $\mathcal{E}$ induced by $\mathcal{D}$.

   *Answer.* Using the formula for the natural extension for a set of desirable gambles, we obtain

$$\begin{aligned}\mathcal{E} =&\{g : \ \forall \delta > 0 \ \exists \lambda_1, \lambda_2, \lambda_3 \geq 0 \text{ s.t.} \\ & g \geq \lambda_1 f_1 + \lambda_2 + \lambda_2 + \lambda_3 f_3 - \delta\};\end{aligned}$$

   if we denote every gamble $f$ as the vector $(f(1), f(2), f(3))$, we obtain that

$$\begin{aligned}\mathcal{E} =&\{g : \ \forall \delta > 0 \exists \lambda_1, \lambda_2, \lambda_3 \geq 0 \text{ s.t.} \\ & g \geq (\lambda_1 + \lambda_3), (\lambda_2 - \lambda_1), (-\lambda_2 - \lambda_3) - \delta\};\end{aligned}$$

   now, note that we can assume without loss of generality that $\lambda_3 = 0$, by considering $\lambda'_1 = \lambda_1 + \lambda_3, \lambda'_2 = \lambda_2 + \lambda_3, \lambda'_3 = 0$; this allows us to express the set $\mathcal{E}$ as

$$\mathcal{E} = \{g : \ \forall \delta > 0 \exists \lambda_1, \lambda_2 \geq 0 \text{ s.t. } g \geq (\lambda_1, \lambda_2 - \lambda_1, -\lambda_2) - \delta\};$$

   finally, remark that we can also make $\delta = 0$ in the above equation, because the set of gambles $\mathcal{D}' := \{(\lambda_1, \lambda_2 - \lambda_1, -\lambda_2) : \lambda_1, \lambda_2 \geq 0\} = \{(\alpha_1, \alpha_2, \alpha_3) : \alpha_1 \geq 0, \alpha_3 \leq 0, \alpha_1 + \alpha_2 + \alpha_3 = 0\}$ is closed in the Euclidean topology, and as a consequence if $g + \delta$ dominates a gamble in this set for every $\delta > 0$ then so does $g$. This implies that

$$\mathcal{E} = \{g : \ \exists \lambda_1, \lambda_2 \geq 0 \text{ s.t. } g \geq (\lambda_1, \lambda_2 - \lambda_1, -\lambda_2)\}.$$

□

   (b) Determine the set $\mathcal{M}$ and the lower prevision $\underline{P}$ induced by $\mathcal{E}$.

*Answer.* Using the correspondence between coherent sets of desirable gambles and sets of linear previsions, we obtain that

$$\mathcal{M}_{\mathcal{E}} := \{P : P(g) \geq 0 \ \forall g \in \mathcal{E}\}.$$

This means that $\mathcal{M}_{\mathcal{E}}$ is the set of linear previsions $P$ satisfying

$$P(1)\lambda_1 + P(2)(\lambda_2 - \lambda_1) + P(3)(-\lambda_2) \geq 0 \ \forall \lambda_1, \lambda_2 \geq 0,$$

or, equivalently,

$$\lambda_1(P(1) - P(2)) + \lambda_2(P(2) - P(3)) \geq 0 \ \forall \lambda_1, \lambda_2 \geq 0;$$

as a consequence,

$$\mathcal{M}_{\mathcal{E}} := \{P : P(1) \geq P(2) \geq P(3)\},$$

and the lower prevision $\underline{P}$ induced by $\mathcal{E}$ is the lower envelope of $\mathcal{M}_{\mathcal{E}}$. □

(c) What is the lower prevision of the gamble $(0, 1, 2)$?

*Answer.* For every prevision $P$ in $\mathcal{M}_{\mathcal{E}}$, we have that

$$P(f) = f(1)P(1) + f(2)P(2) + f(3)P(3) = P(2) + 2P(3);$$

if we take the prevision $P$ given by $P(1) = 1, P(2) = P(3) = 0$, we get $P(f) = 0$, and since this $P$ belongs to $\mathcal{M}_{\mathcal{E}}$ we deduce that $\underline{P}(f) \leq P(f) = 0$; but since from coherence $\underline{P}(f) \geq \min f = 0$, we deduce that in this case $\underline{P}(f) = 0$. □