Imprecise Probabilistic Graphical Models

Credal Networks and Other Guys

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Dalle Molle Institute for Artificial Intelligence IDSIA

SISPTA School on Imprecise Probability

Durham, September 5, 2010

Imprecise Probability Group @ IDSIA

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- IDSIA = *Dalle Molle* Institute for AI
- Imprecise Probability Group (1 professor, 4 researchers, 1 phd)
  - Theory of imprecise probability
  - Probabilistic graphical models
  - Data mining and classification
  - Observations modelling (missing data)
  - Data fusion and filtering
  - Applications to environmental modelling, military decision making, risk analysis, bioinformatics, biology, tracking, vision, . . .

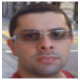
Marco  Zaffalon

Alex  Antonucci

Cassio  de Campos

Giorgio  Corani

Alessio  Benavoli

Denis  Mauà

Probabilistic Graphical Models

aka **Decomposable** Multivariate Probabilistic Models

(whose decomposability is induced by **independence**)

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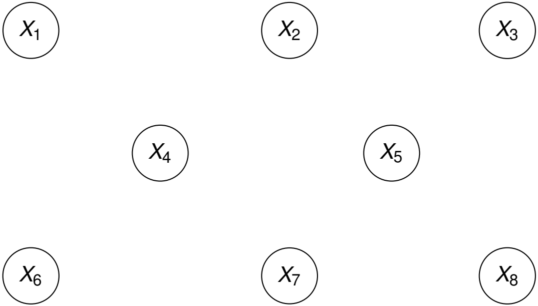

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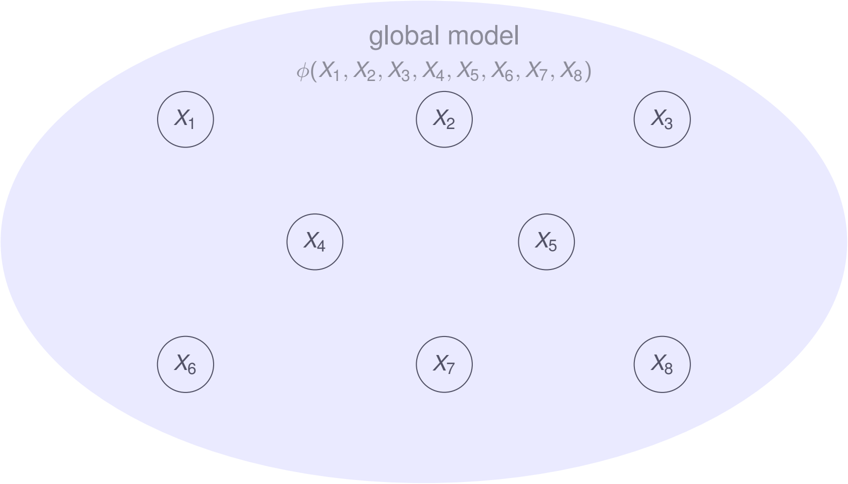

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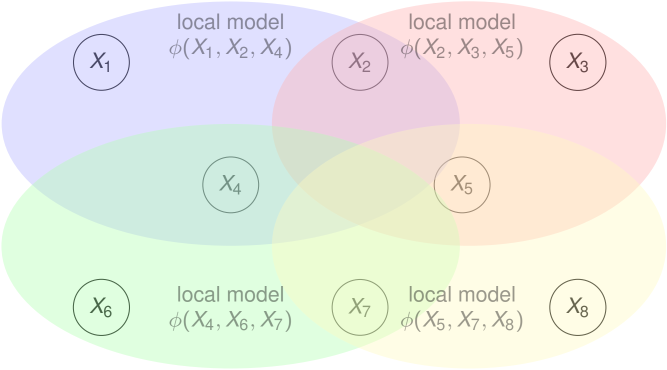
$$\phi(X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8) = \phi(X_1, X_2, X_4) \otimes \phi(X_2, X_3, X_5) \otimes \phi(X_4, X_6, X_7) \otimes \phi(X_5, X_7, X_8)$$



Probabilistic Graphical Models

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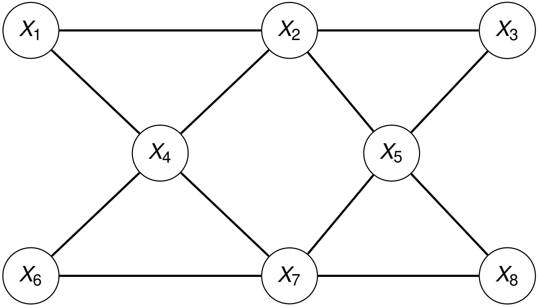
undirected graphs

precise/imprecise Markov random fields



Probabilistic Graphical Models

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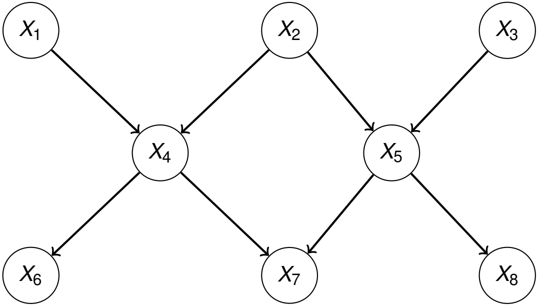
directed graphs

Bayesian/credal networks



Probabilistic Graphical Models

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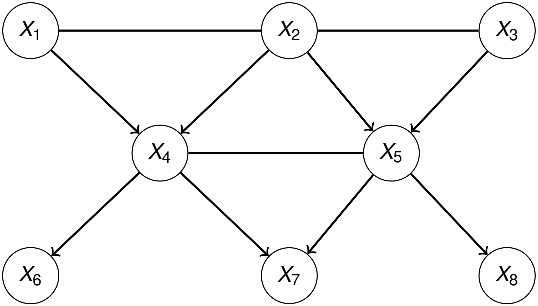
mixed graphs

chain graphs



[Exe #1] Fault trees (Vesely et al, 1981)

brake fails = [pads $\vee$ (sensor $\wedge$ controller $\wedge$ actuator)]

devices failures are independent

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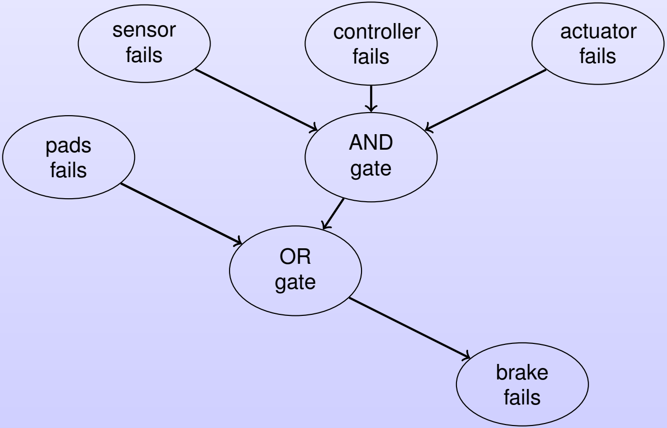

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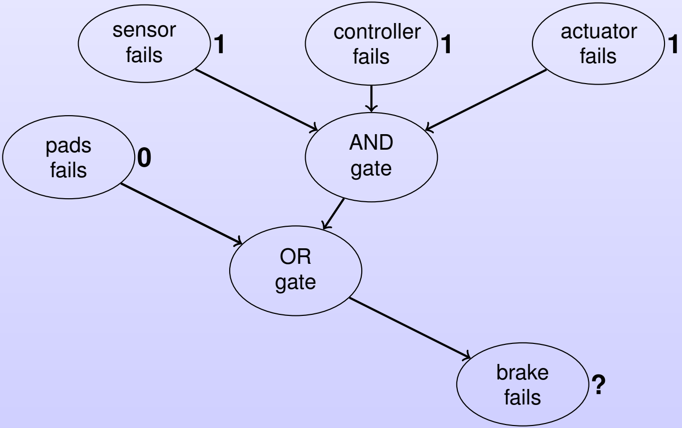

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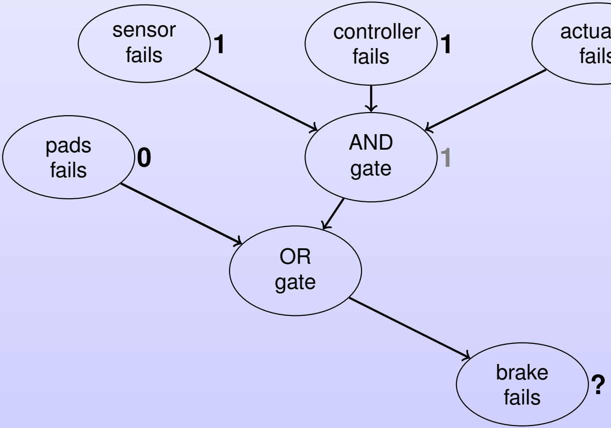

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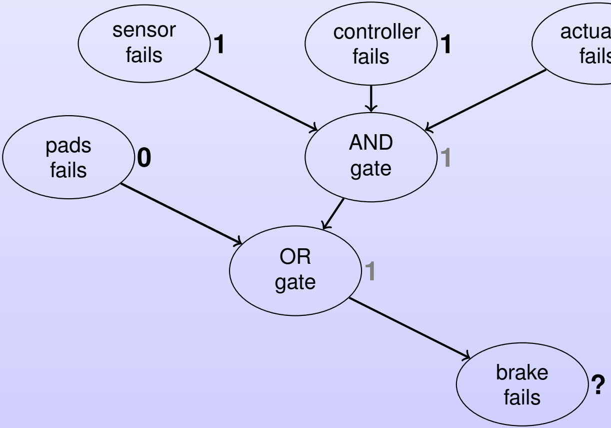

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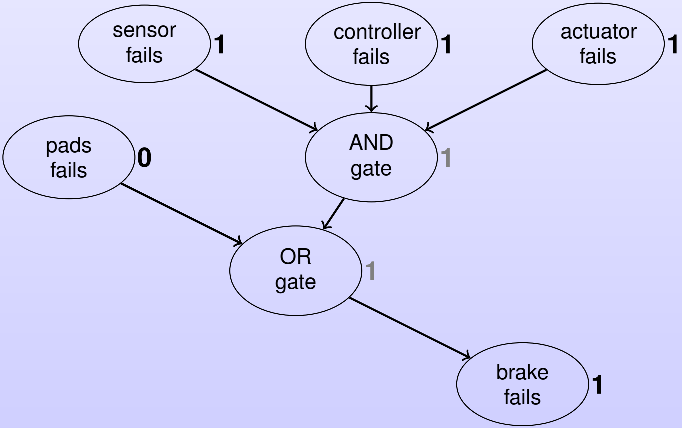

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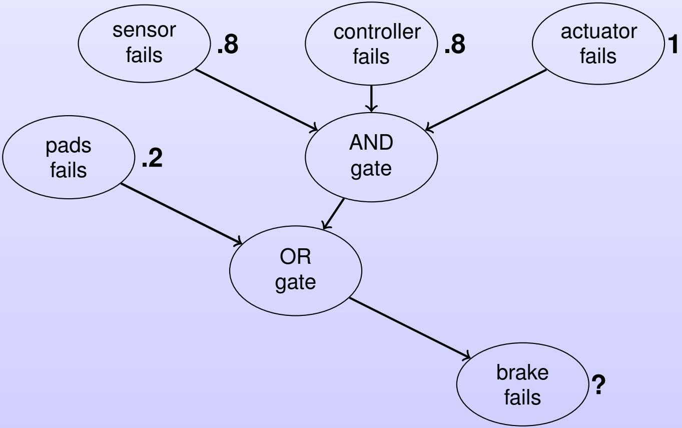

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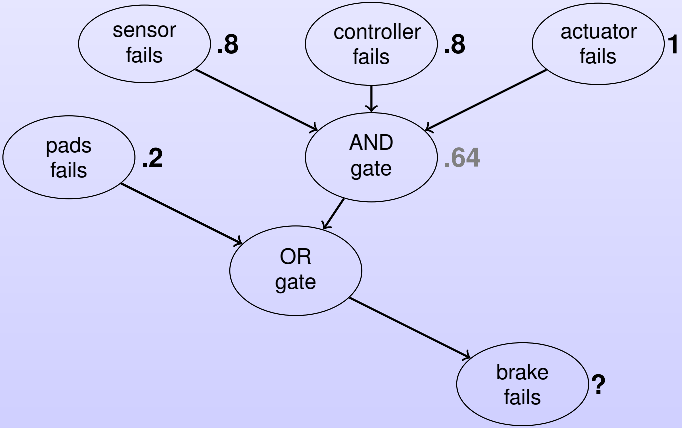

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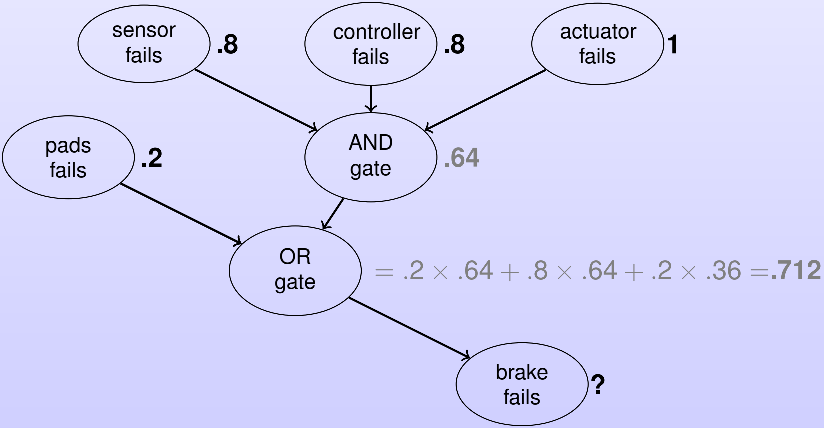

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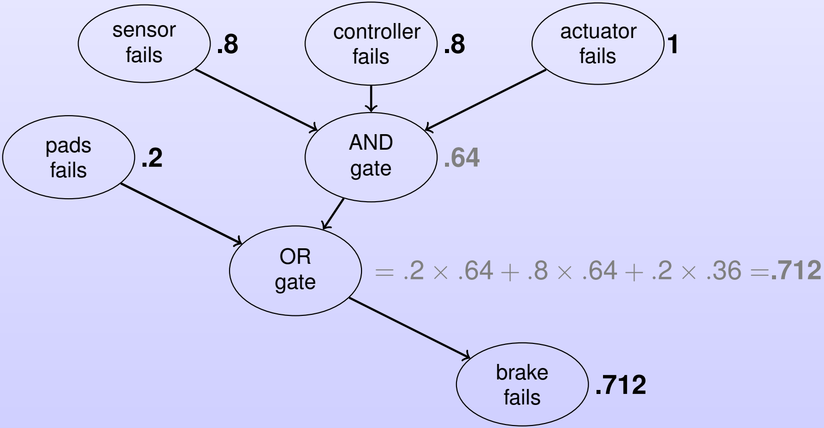

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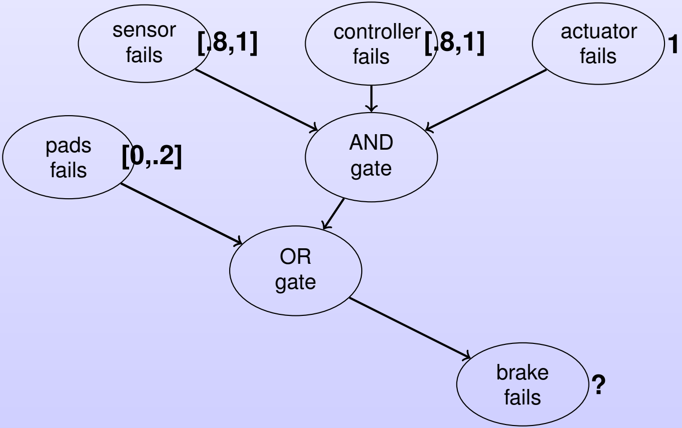

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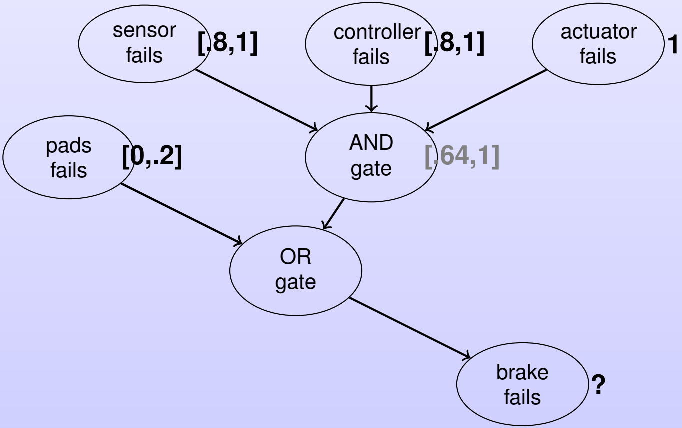

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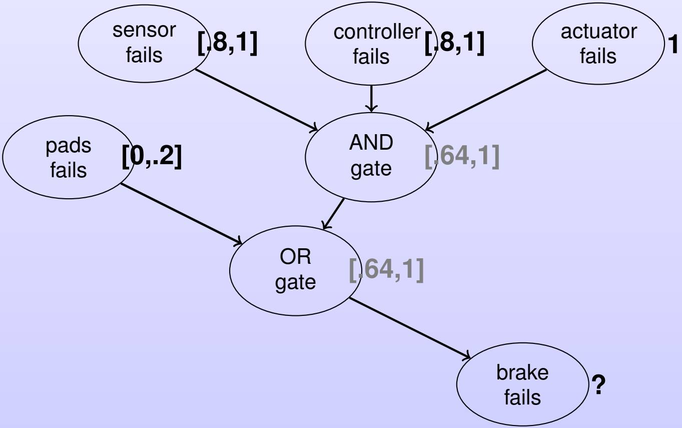

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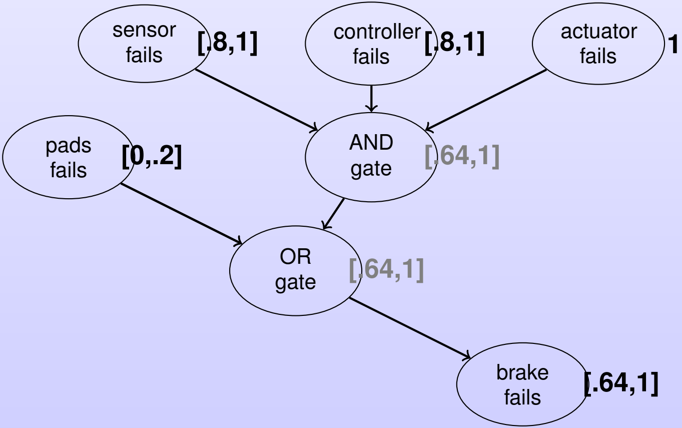

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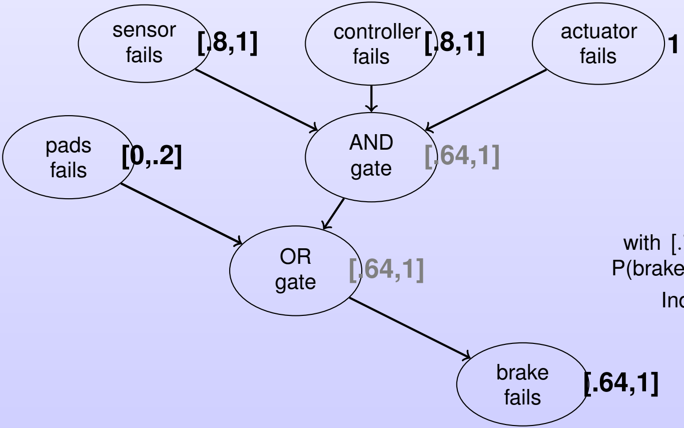

with [.7, 1] instead
P(brake fails)∈[.49,1]

Indecision!

Outline

- Motivations for imprecise probability
- Credal sets (basic concepts and operations)
- Independence relations
- Credal networks
- Modelling observations/missingness
- Decision making
- Inference algorithms
- Other probabilistic graphical models
- Conclusions

Three different levels of knowledge

- FIFA'10 final match between Holland and Spain
- Result of Holland after the regular time? Win, draw or loss?

DETERMINISM

The Dutch goalkeeper is unbeatable and Holland always makes a goal

Holland (certainly) wins

$$\begin{matrix} P(Win) \\ P(Draw) \\ P(Loss) \end{matrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

UNCERTAINTY

Win is two times more probable than draw, and this being three times more probable than loss

$$\begin{matrix} P(Win) \\ P(Draw) \\ P(Loss) \end{matrix} = \begin{bmatrix} .6 \\ .3 \\ .1 \end{bmatrix}$$

IMPRECISION

Win is more probable than draw, and this is more probable than loss

$P(Win) > P(Draw)$
$P(Draw) > P(Loss)$

$$\begin{matrix} P(Win) \\ P(Draw) \\ P(Loss) \end{matrix} = \begin{bmatrix} \frac{\alpha}{3} + \beta + \frac{\gamma}{2} \\ \frac{\alpha}{3} + \frac{\gamma}{2} \\ \frac{\alpha}{3} \end{bmatrix}$$

$\forall \alpha, \beta, \gamma$ such that
$\alpha > 0, \beta > 0, \gamma > 0,$
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Three different levels of knowledge (ii)

DETERMINISM

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IMPRECISION

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DETERMINISM UNCERTAINTY IMPRECISION

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Limit when learning from large (complete) data sets

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Propositional (Boolean) Logic

Bayesian Theory of Probability

Walley's Theory of Coherent Lower Previsions

Three different levels of knowledge (ii)

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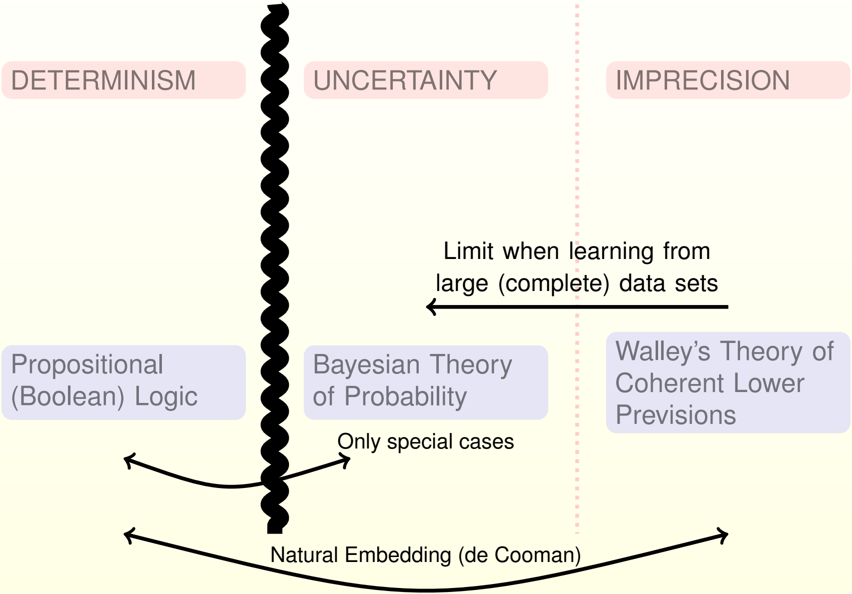
Natural Embedding (de Cooman)

Three different levels of knowledge (ii)



Three different levels of knowledge (ii)

DETERMINISM

Propositional (Boolean) Logic

UNCERTAINTY

Bayesian Theory of Probability

Only special cases

IMPRECISION

Small/incomplete data

Expert's (qualitative) knowledge

Vague/incomplete observations

Limit when learning from large (complete) data sets

Walley's Theory of Coherent Lower Previsions

Natural Embedding (de Cooman)

From CLPs to Credal Sets

Modelling knowledge about X, taking values in $\mathcal{X}$

From CLPs to Credal Sets

Modelling knowledge about X, taking values in $\mathcal{X}$

Bayesian / precise

probability distribution
$p : \mathcal{X} \to \mathbb{R}$

$$\begin{cases} p(x) \geq 0 \forall x \in \mathcal{X} \\ \sum_{x \in \mathcal{X}} p(x) = 1 \end{cases}$$

coherent **linear** prevision
$P : \mathcal{L}(\mathcal{X}) \to \mathbb{R}$

$$\begin{cases} P(f+g) = P(f) + P(g) \\ P(f) \geq \inf f, \forall f, g \in \mathcal{L}(\mathcal{X}) \end{cases}$$

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⟷

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$\Longleftrightarrow$

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Credal / imprecise

set of distributions
$K(X) = \{\ p(X) \mid \mathit{constraints}\ \}$

coherent lower prevision
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$\exists$ set $\mathcal{M}$ of linear previsions s.t.
$\underline{P}(f) = \inf_{P \in \mathcal{M}} P(f) \; \forall f \in \mathcal{L}(\mathcal{X})$

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lower envelope Theorem

(a set and its convex closure have the same lower envelope!)

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Credal / imprecise

set of distributions **convex closed**
$K(X) = \{ \ p(X) \mid \textit{constraints} \ \}$

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lower envelope Theorem

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Credal sets (Levi, 1980)

- A closed convex set $K(X)$ of probability mass functions
- Equivalently described by its *extreme points* $\mathrm{ext}[K(X)]$
- Focus on categorical variables ($|\mathcal{X}| < +\infty$) and finitely generated CSs (polytopes, $|\mathrm{ext}[K(X)]| < +\infty$)

V-representation	⟷	H-representation
enumerate the	LRS software	linear constraints
extreme points	(Avis & Fukuda)	on the probabilities

- Given a function (gamble) $f(X)$, lower expectation: $\underline{E}[f(X)] := \min_{P(X)\in K(X)} \sum_{x\in\mathcal{X}} P(x)f(x)$
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Credal sets (Levi, 1980)

- A closed convex set $K(X)$ of probability mass functions
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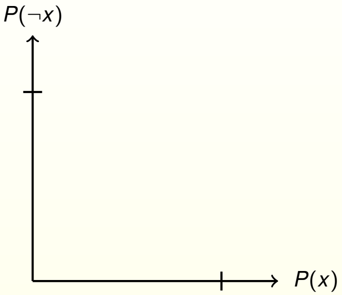

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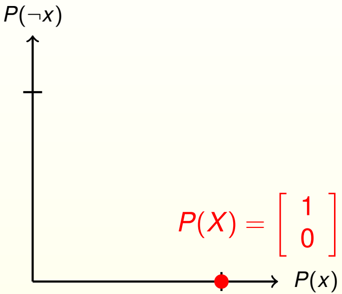

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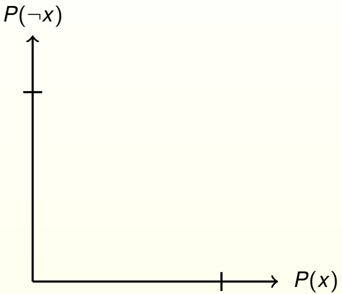

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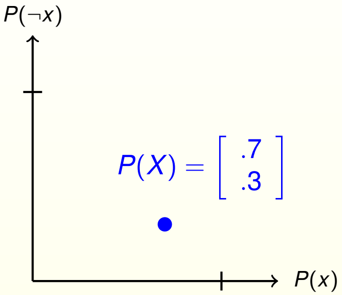

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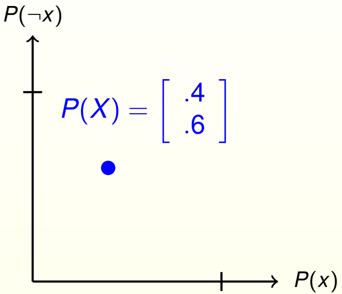

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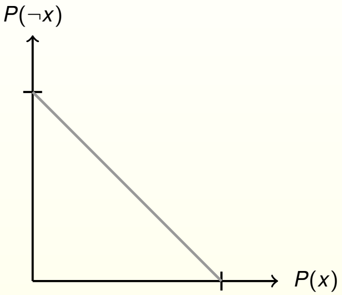

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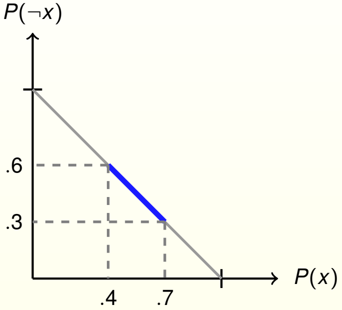
$$K(X) \equiv \left\{ P(X) = \begin{bmatrix} p \\ 1-p \end{bmatrix} \middle| .4 \leq p \leq .7 \right\}$$



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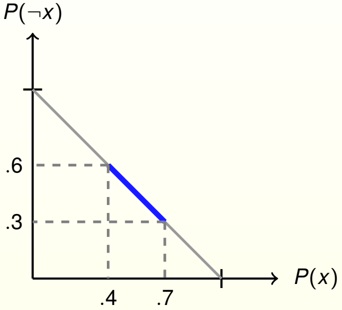
- A CS over a Boolean variable cannot have more than two vertices!

$$\mathrm{ext}[K(X)] = \left\{ \begin{bmatrix} .7 \\ .3 \end{bmatrix}, \begin{bmatrix} .4 \\ .6 \end{bmatrix} \right\}$$



Geometric Representation of CSs (ternary variables)

- Ternary X (e.g., $\mathcal{X} = \{\text{win,draw,loss}\}$)
- $P(X) \equiv$ point in the space (simplex)
- No bounds to $|\text{ext}[K(X)]|$
- Modelling ignorance
  - Uniform models indifference
  - Vacuous credal set
- Expert qualitative knowledge
  - Win is more probable than draw, which more probable than loss
- Learning from small datasets
  - Learning with multiple priors (e.g., IDM with $s = 2$)
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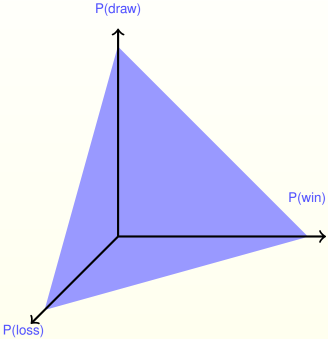

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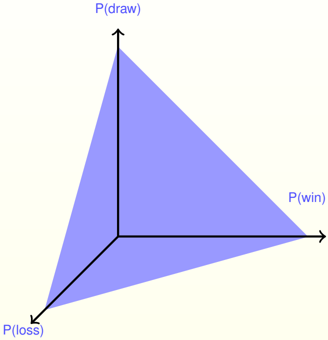

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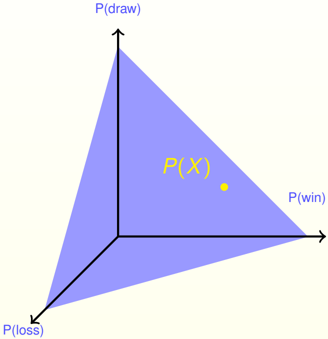

$$P(X) = \begin{bmatrix} .6 \\ .3 \\ .1 \end{bmatrix}$$

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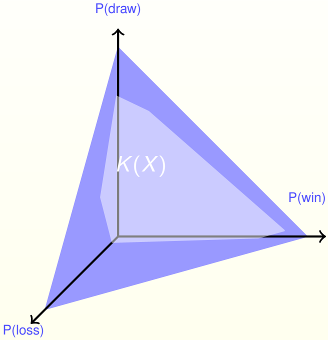

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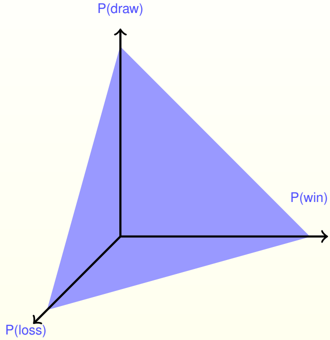

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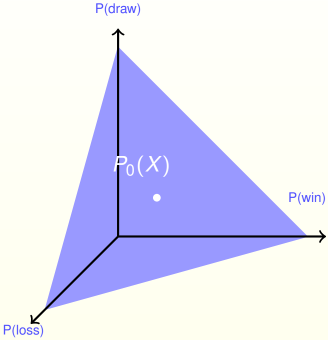

$P_0(x) = \frac{1}{|\Omega_X|}$

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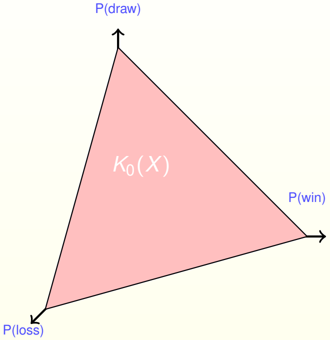

$$K_0(X) = \left\{ P(X) \;\middle|\; \begin{array}{l} \sum_x P(x) = 1, \\ P(x) \geq 0 \end{array} \right\}$$

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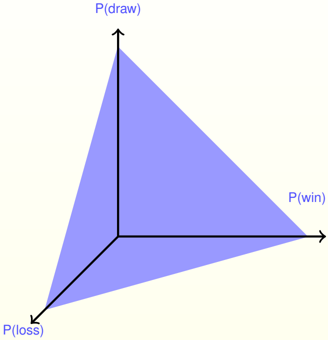

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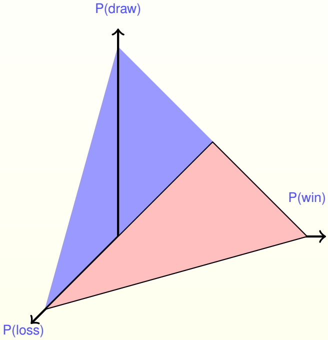

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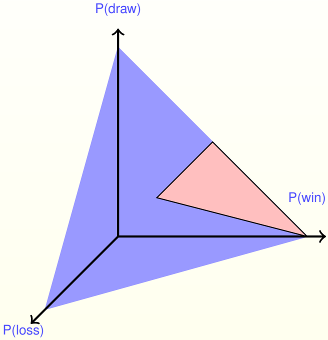

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From natural language to linear contraints on probabilities

(Walley, 1991)

extremely probable $P(x) \geq 0.98$
very high probability $P(x) \geq 0.9$
highly probable $P(x) \geq 0.85$
very probable $P(x) \geq 0.75$
has a very good chance $P(x) \geq 0.65$
quite probable $P(x) \geq 0.6$
$P(x) \geq 0.5$
has a good chance $0.4 \leq P(x) \leq 0.85$
is improbable (unlikely) $P(x) \leq 0.5$
is somewhat unlikely $P(x) \leq 0.4$
is very unlikely $P(x) \leq 0.25$
has little chance $P(x) \leq 0.2$
is highly improbable $P(x) \leq 0.15$
is has very low probability $P(x) \leq 0.1$
is extremely unlikely $P(x) \leq 0.02$

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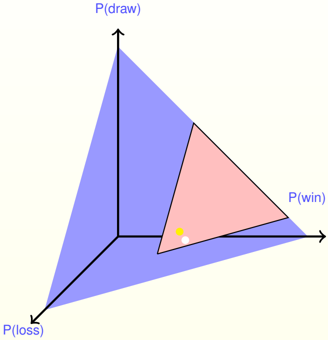

Previous matches:
Holland 4 wins,
Draws 1,
Spain 3 wins

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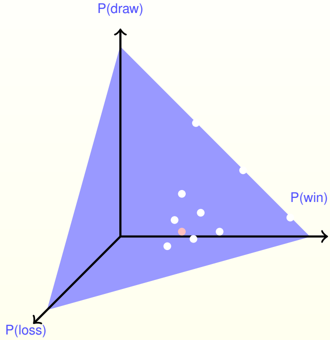

1957:	Spain	vs.	Holland	5 — 1	
1973:	Holland	vs.	Spain	3 — 2	
1980:	Spain	vs.	Holland	1 — 0	
1983:	Spain	vs.	Holland	1 — 0	
1983:	Holland	vs.	Spain	2 — 1	
1987:	Spain	vs.	Holland	1 — 1	
2000:	Spain	vs.	Holland	1 — 2	
2001:	Holland	vs.	Spain	1 — 0	
2005:	Spain	vs.	Holland	$*$ — $*$	(missing)
2008:	Holland	vs.	Spain	$*$ — $*$	(missing)

Basic operations with credal sets

	PRECISE Mass functions	IMPRECISE Credal sets
Joint	$P(X,Y)$	$K(X,Y)$
Marginalization	$P(X)$ s.t. $p(x) = \sum_y p(x,y)$	$K(X) = \left\{ P(X) \,\middle\vert\, \begin{array}{l} P(x) = \sum_y P(x,y) \\ P(X,Y) \in K(X,Y) \end{array} \right\}$
Conditioning	$P(X\|y)$ s.t. $p(x\|y) = \frac{P(x,y)}{\sum_y P(x,y)}$	$K(X\|y) = \left\{ P(X\|y) \,\middle\vert\, \begin{array}{l} P(x\|y) = \frac{P(x,y)}{\sum_y P(x,y)} \\ P(X,Y) \in K(X,Y) \end{array} \right\}$
Combination	$P(x,y) = P(x\|y)P(y)$	$K(X\|Y) \otimes K(Y) = \left\{ P(X,Y) \,\middle\vert\, \begin{array}{l} P(x,y)=P(x\|y)P(y) \\ P(X\|y) \in K(X\|y) \\ P(Y) \in K(Y) \end{array} \right\}$

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Joint	$P(X,Y)$	$K(X,Y)$
Marginalization	$P(X)$ s.t. $p(x) = \sum_y p(x,y)$	$K(X) = \left\{ P(X) \,\middle\vert\, \begin{array}{l} P(x) = \sum_y P(x,y) \\ P(X,Y) \in K(X,Y) \end{array} \right\}$
Conditioning	$P(X\|y)$ s.t. $p(x\|y) = \frac{P(x,y)}{\sum_y P(x,y)}$	$K(X\|y) = \left\{ P(X\|y) \,\middle\vert\, \begin{array}{l} P(x\|y) = \frac{P(x,y)}{\sum_y P(x,y)} \\ P(X,Y) \in K(X,Y) \end{array} \right\}$
Combination	$P(x,y) = P(x\|y)P(y)$	$K(X\|Y) \otimes K(Y) = \left\{ P(X,Y) \,\middle\vert\, \begin{array}{l} P(x,y)=P(x\|y)P(y) \\ P(X\|y) \in K(X\|y) \\ P(Y) \in K(Y) \end{array} \right\}$

Basic operations with credal sets

	PRECISE Mass functions	IMPRECISE Credal sets
Joint	$P(X, Y)$	$K(X, Y)$
Marginalization	$P(X)$ s.t. $p(x) = \sum_y p(x, y)$	$K(X) = \left\{ P(X) \,\middle\vert\, \begin{array}{l} P(x) = \sum_y P(x, y) \\ P(X, Y) \in K(X, Y) \end{array} \right\}$
Conditioning	$P(X\|y)$ s.t. $p(x\|y) = \frac{P(x,y)}{\sum_y P(x,y)}$	$K(X\|y) = \left\{ P(X\|y) \,\middle\vert\, \begin{array}{l} P(x\|y) = \frac{P(x,y)}{\sum_y P(x,y)} \\ P(X, Y) \in K(X, Y) \end{array} \right\}$
Combination	$P(x, y) = P(x\|y)P(y)$	$K(X\|Y) \otimes K(Y) = \left\{ P(X, Y) \,\middle\vert\, \begin{array}{l} P(x, y) = P(x\|y)P(y) \\ P(X\|y) \in K(X\|y) \\ P(Y) \in K(Y) \end{array} \right\}$

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Basic operations with credal sets (vertices)

	IMPRECISE Credal sets	IMPRECISE Extremes
Joint	$K(X,Y)$	
Marginalization	$K(X) = \left\{ P(X) \;\middle|\; \begin{array}{l} P(x) = \sum_y P(x,y) \\ P(X,Y) \in K(X,Y) \end{array} \right\}$	
Conditioning	$K(X\|y) = \left\{ P(X\|y) \;\middle|\; \begin{array}{l} P(x\|y) = \frac{P(x,y)}{\sum_y P(x,y)} \\ P(X,Y) \in K(X,Y) \end{array} \right\}$	
Combination	$K(X\|Y) \otimes K(Y) = \left\{ P(X,Y) \;\middle|\; \begin{array}{l} P(x,y)=P(x\|y)P(y) \\ P(X\|y) \in K(X\|y) \\ P(Y) \in K(Y) \end{array} \right\}$	

Basic operations with credal sets (vertices)

	IMPRECISE Credal sets	IMPRECISE Extremes
Joint	$K(X,Y)$	$= \mathrm{CH}\left\{P_j(X,Y)\right\}_{j=1}^{n_v}$
Marginalization	$K(X) = \left\{P(X) \,\middle\vert\, \begin{array}{l} P(x)=\sum_y P(x,y) \\ P(X,Y)\in K(X,Y)\end{array}\right\}$	$= \mathrm{CH} \left\{P(X) \,\middle\vert\, \begin{array}{l} P(x)=\sum_y P(x,y) \\ P(X,Y)\in \mathrm{ext}[K(X,Y)]\end{array}\right\}$
Conditioning	$K(X\vert y) = \left\{P(X\vert y) \,\middle\vert\, \begin{array}{l} P(x\vert y)=\frac{P(x,y)}{\sum_y P(x,y)} \\ P(X,Y)\in K(X,Y)\end{array}\right\}$	$= \mathrm{CH} \left\{P(X\vert y) \,\middle\vert\, \begin{array}{l} P(x\vert y)=\frac{P(x,y)}{\sum_y P(x,y)} \\ P(X,Y)\in \mathrm{ext}[K(X,Y)]\end{array}\right\}$
Combination	$K(X\vert Y)\otimes K(Y) = \left\{P(X,Y) \,\middle\vert\, \begin{array}{l} P(x,y)=P(x\vert y)P(y) \\ P(X\vert y)\in K(X\vert y) \\ P(Y)\in K(Y)\end{array}\right\}$	$= \mathrm{CH} \left\{P(X,Y) \,\middle\vert\, \begin{array}{l} P(x,y)=P(x\vert y)P(y) \\ P(X\vert y)\in \mathrm{ext}[K(X\vert y)] \\ P(Y)\in \mathrm{ext}[K(Y)]\end{array}\right\}$

Basic operations with credal sets (vertices)

	IMPRECISE Credal sets	IMPRECISE Extremes
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Marginalization	$K(X) = \left\{P(X) \,\middle\vert\, \begin{array}{l} P(x)=\sum_y P(x,y) \\ P(X,Y)\in K(X,Y)\end{array}\right\}$	$= \mathrm{CH} \left\{P(X) \,\middle\vert\, \begin{array}{l} P(x)=\sum_y P(x,y) \\ P(X,Y)\in \mathrm{ext}[K(X,Y)]\end{array}\right\}$
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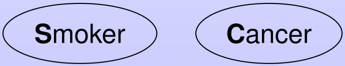
[EXE]
Prove it!

[Exe #2] An imprecise bivariate (graphical?) model

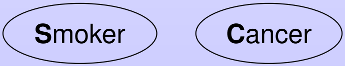
- Two Boolean variables: **S**moker, Lung **C**ancer



[Exe #2] An imprecise bivariate (graphical?) model

- Two Boolean variables: **S**moker, Lung **C**ancer
- Eight "Bayesian" phisicians, each one assessing $P_j(S, C)$

j	$P_j(s, c)$	$P_j(s, \neg c)$	$P_j(\neg s, c)$	$P_j(\neg s, \neg c)$
1	1/8	1/8	3/8	3/8
2	1/8	1/8	9/16	3/16
3	3/16	1/16	3/8	3/8
4	3/16	1/16	9/16	3/16
5	1/4	1/4	1/4	1/4
6	1/4	1/4	3/8	1/8
7	3/8	1/8	1/4	1/4
8	3/8	1/8	3/8	1/8



[Exe #2] An imprecise bivariate (graphical?) model

- Two Boolean variables: **S**moker, Lung **C**ancer
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- $K(S,C) = \mathrm{CH}\left\{P_j(S,C)\right\}_{j=1}^{8}$

j	$P_j(s,c)$	$P_j(s,\neg c)$	$P_j(\neg s,c)$	$P_j(\neg s,\neg c)$
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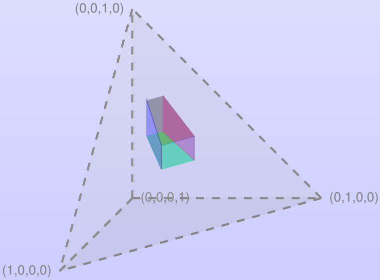
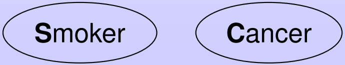
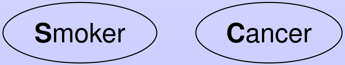




[Exe #2] An imprecise bivariate (graphical?) model

- Two Boolean variables: **S**moker, Lung **C**ancer
- Eight “Bayesian” phisicians, each one assessing $P_j(S, C)$
- $K(S, C) = \mathrm{CH}\left\{P_j(S, C)\right\}_{j=1}^{8}$
- Compute:
  - Marginal $K(S)$
  - Conditioning $K(C|S) := \{K(C|s), K(C|s)\}$
  - Combination (marg ext) $K'(C, S) := K(C|S) \otimes K(S)$



j	$P_j(s, c)$	$P_j(s, \neg c)$	$P_j(\neg s, c)$	$P_j(\neg s, \neg c)$
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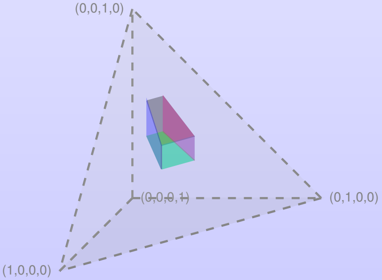

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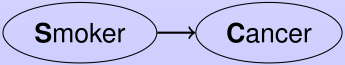
  - Combination (marg ext) $K'(C,S) := K(C|S) \otimes K(S)$
- Is this a (I)PGM?



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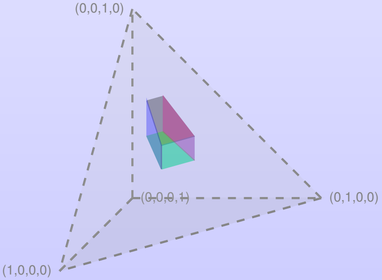

Cano-Cano-Moral Transformation

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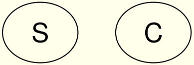
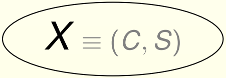


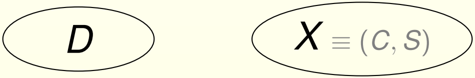
Cano-Cano-Moral Transformation

- Joint variable $X := (C, S)$, $K(X) = \{P_j(X)\}_{j=1}^{n_v}$ $(|\mathcal{X} = 4|$ and $n_v = 8)$



Cano-Cano-Moral Transformation

- Joint variable $X := (C, S)$, $K(X) = \{P_j(X)\}_{j=1}^{n_v}$ ($|\mathcal{X} = 4|$ and $n_v = 8$)
- Auxiliary variable D to enumerate elements of $\mathrm{ext}[K(X)]$



Cano-Cano-Moral Transformation

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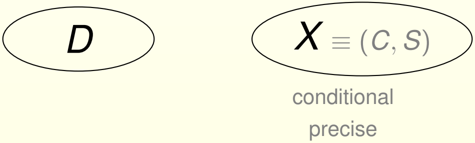
  - Precise $K(X|D)$, with $P(X|d_j) := P_j(X)$ with $|\mathcal{D}| = 8$



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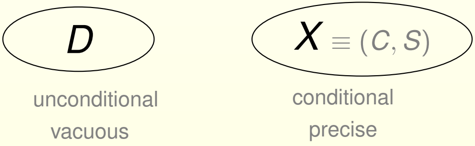
  - Precise $K(X|D)$, with $P(X|d_j) := P_j(X)$ with $|\mathcal{D}| = 8$
  - Vacuous $K(D)$ (8 vertices)



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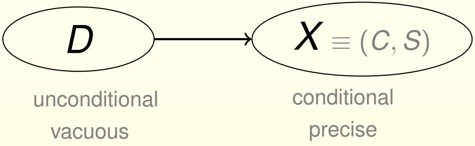
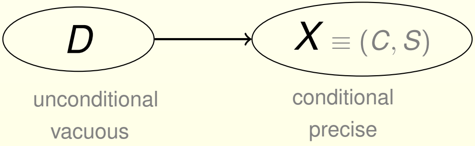
  - Precise $K(X|D)$, with $P(X|d_j) := P_j(X)$ with $|\mathcal{D}| = 8$
  - Vacuous $K(D)$ (8 vertices)
- Marginal extension $K(X, D) = K(X|D) \otimes K(D)$



Cano-Cano-Moral Transformation

- Joint variable $X := (C, S)$, $K(X) = \{P_j(X)\}_{j=1}^{n_v}$ ($|\mathcal{X} = 4|$ and $n_v = 8$)
- Auxiliary variable D to enumerate elements of $\mathrm{ext}[K(X)]$
  - Precise $K(X|D)$, with $P(X|d_j) := P_j(X)$ with $|\mathcal{D}| = 8$
  - Vacuous $K(D)$ (8 vertices)
- Marginal extension $K(X, D) = K(X|D) \otimes K(D)$
- After marginalization of D, $K(X, D)$ becomes $K(X)$



Cano-Cano-Moral Transformation

- Joint variable $X := (C, S)$, $K(X) = \{P_j(X)\}_{j=1}^{n_v}$ ($|\mathcal{X} = 4|$ and $n_v = 8$)
- Auxiliary variable D to enumerate elements of $\mathrm{ext}[K(X)]$
  - Precise $K(X|D)$, with $P(X|d_j) := P_j(X)$ with $|\mathcal{D}| = 8$
  - Vacuous $K(D)$ (8 vertices)
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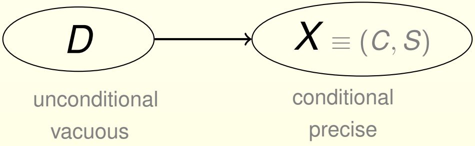

Cano-Cano-Moral Transformation

- Joint variable $X := (C, S)$, $K(X) = \{P_j(X)\}_{j=1}^{n_v}$ ($|\mathcal{X} = 4|$ and $n_v = 8$)
- Auxiliary variable D to enumerate elements of $\text{ext}[K(X)]$
  - Precise $K(X|D)$, with $P(X|d_j) := P_j(X)$ with $|\mathcal{D}| = 8$
  - Vacuous $K(D)$ (8 vertices)
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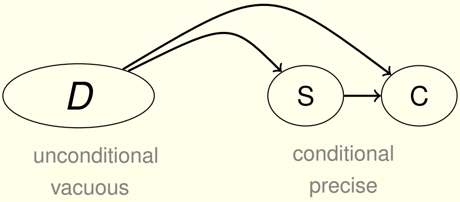

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Stochastic independence/irrelevance (precise case)

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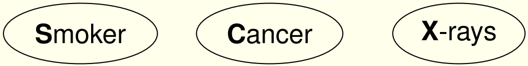
Every notion admits a conditional formulation

Other IP independence concepts (epistemic, Kuznetzov, strict)

A tri-variate example

- 3 Boolean variables: **S**moker, Lung **C**ancer, **X**-rays



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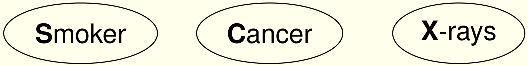

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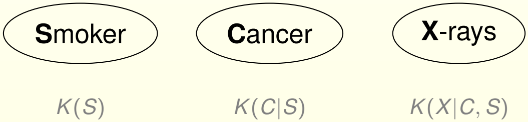

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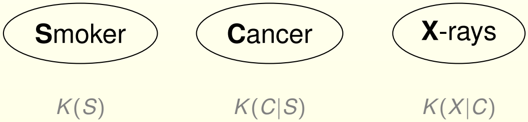

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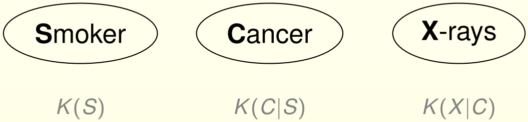

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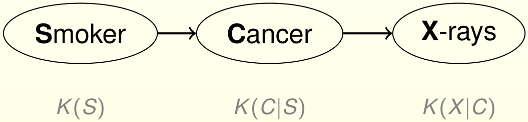

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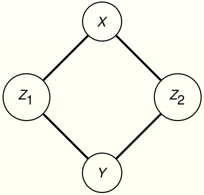

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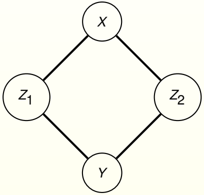

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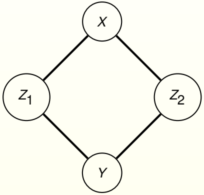

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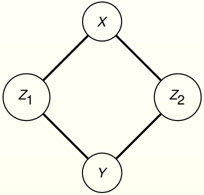

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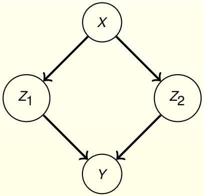

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credal network specification

credal network strong extension

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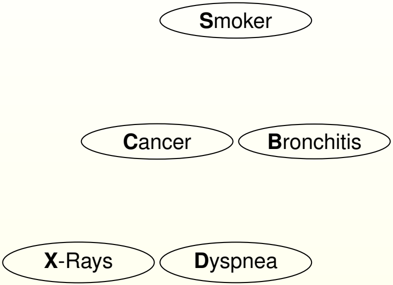

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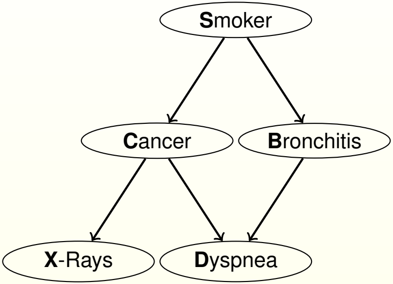

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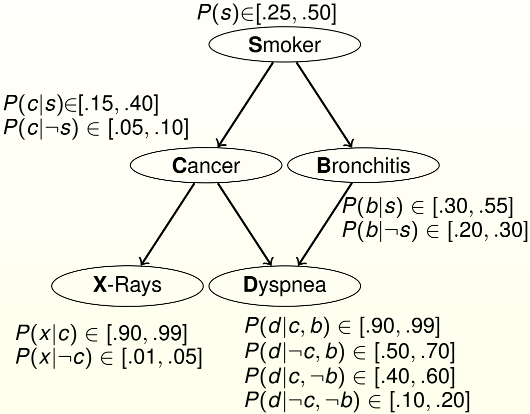

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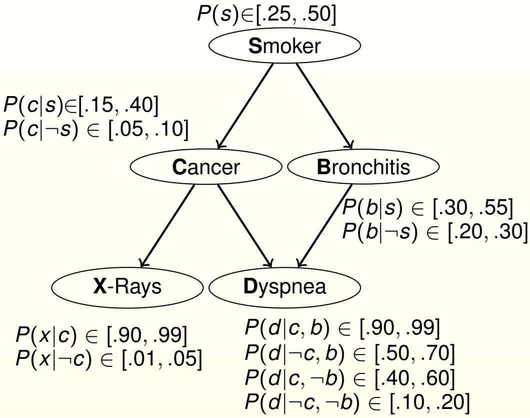

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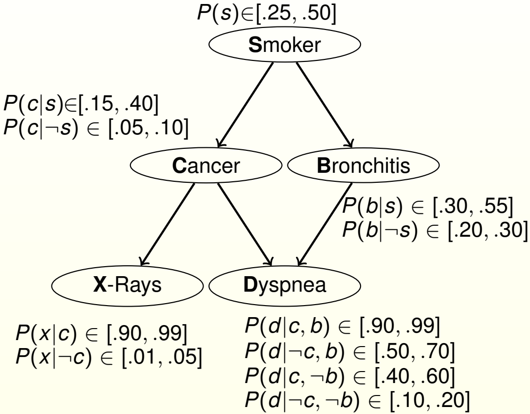

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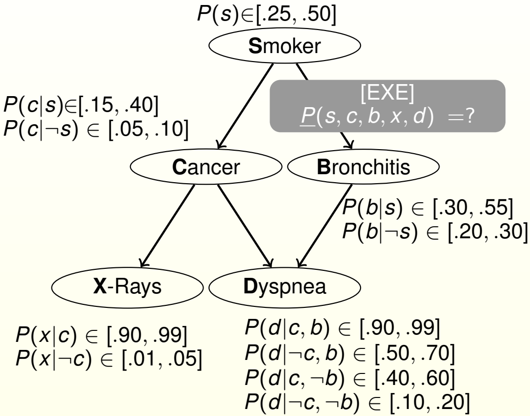

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- The vertices of the SE correspond to Bayesian networks!
- A CN = collection of BN (all with the same graph) *n* exponential
- Sensitivity analysis interpretation
  $\mathrm{ext}[K(X_1, \ldots, X_n)] = \{P_j(X_1, \ldots, X_n)\}_{j=1}^{n}$

Non-separately specified CNs

- Constraints among different conditional mass functions of a CN
- Explicit enumeration of the relative BNs
  - Auxiliary parent selecting the conditional probabilities *(Cano, Cano, Moral, 1994)* with a vacuous prior
- "Extensive" specification
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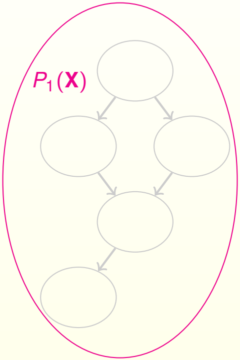

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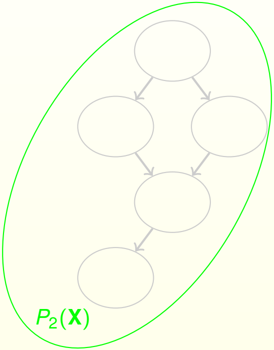

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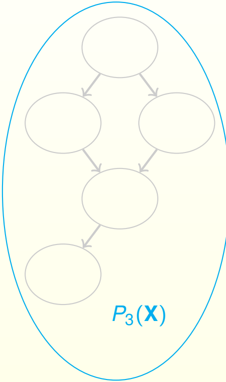

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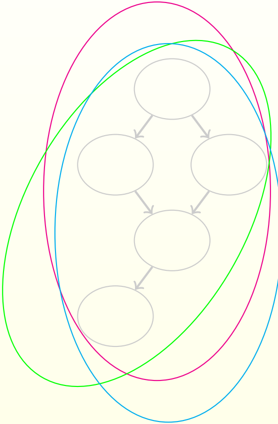

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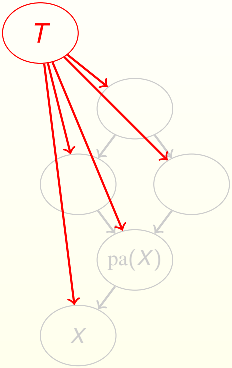

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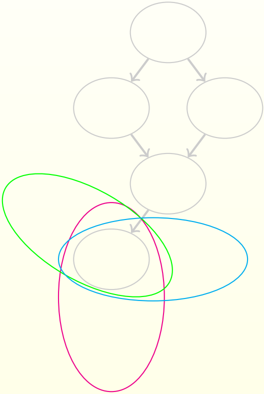

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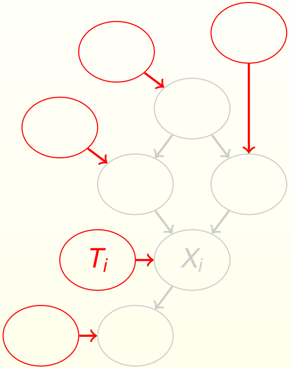

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  - Auxiliary parent selecting the conditional probabilities *(Cano, Cano, Moral, 1994)* with a vacuous prior
- "Extensive" specification
  - Constraints among conditional mass functions of the same variable
  - Each CPT takes values from a set of tables an auxiliary parent selecting the tables
- An unconstrained (i.e., separated) specification is always possible *(Antonucci & Zaffalon, IJAR, 2008)*

Updating credal nets

- A variable of interest X_q (query)
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$$P(x_q|x_E) = \frac{\sum_{x_M \in \Omega_{X_M}} \prod_{i=1}^n P(x_i|\mathrm{pa}(X_i))}{\sum_{x_M \in \Omega_{X_M}, x_q \in \Omega_{X_q}} \prod_{i=1}^n P(x_i|\mathrm{pa}(X_i))}$$

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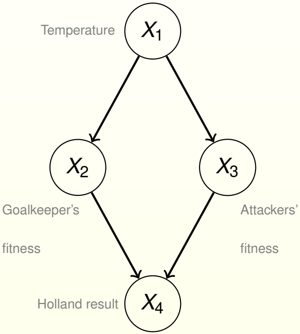

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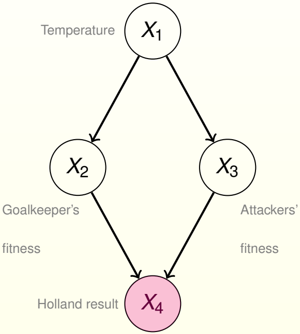

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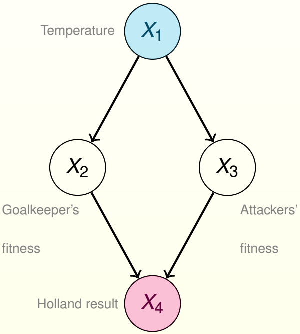

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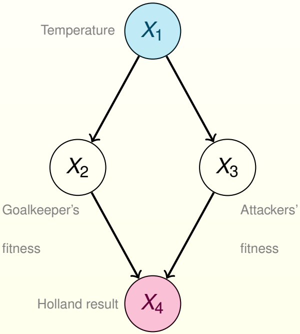

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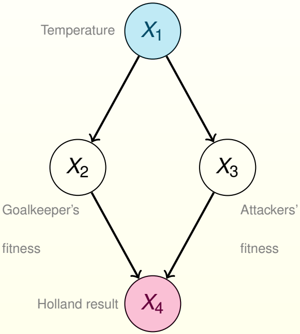

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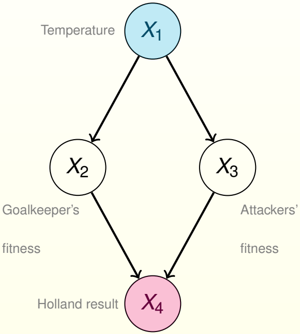

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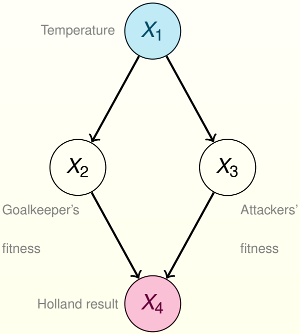

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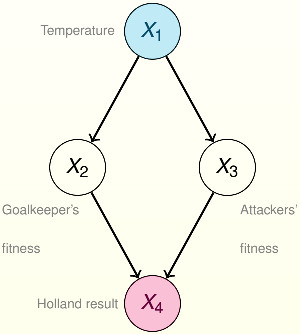

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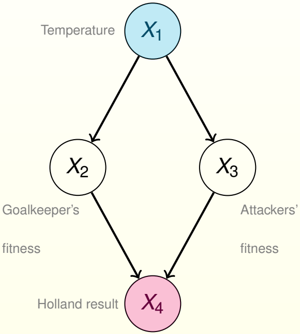

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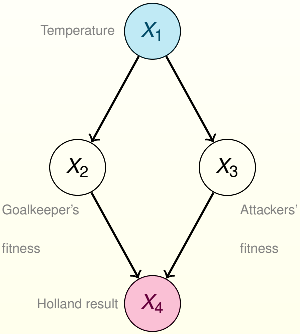

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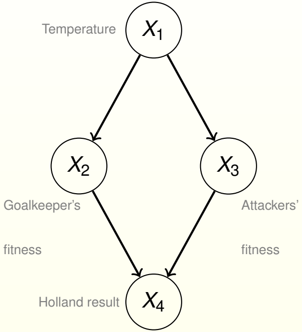
Updating with incomplete observations

- $\underline{P}(X_q = x_q | X_E = x_E, X_M = *)$
  $= \underline{P}(X_q = x_q | X_E = x_E)$
  right only if *missing at random* assumption holds
- *Conservative inference rule* (CIR)
  $\underline{P}(x_q | x_E, *) = \min_{x_M \in \Omega_{X_M}} \underline{P}(x_q | x_E, x_M)$
  near-ignorance about the process preventing some variable from being observed *(de Cooman & Zaffalon, 2004)*
- CIR on CNs?
  - Add a (dummy) binary child for each missing, with vacuous quantification
  - Use standard updating algorithms



$\underline{P}(x_4 | X_1 = warm, X_2 = *)$

Updating with incomplete observations

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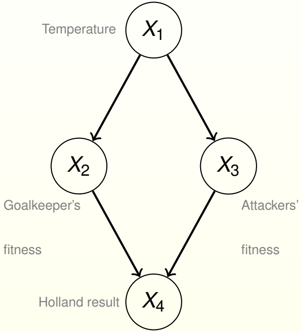

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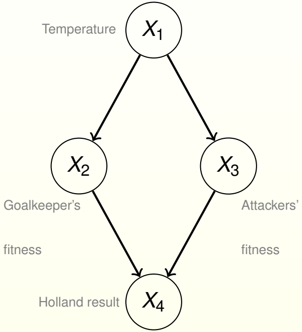

$$\underline{P}(x_4 | X_1 = warm, X_2 = *)$$
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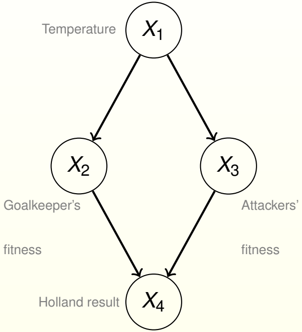

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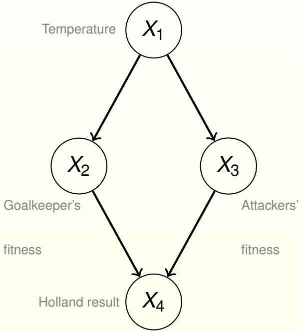

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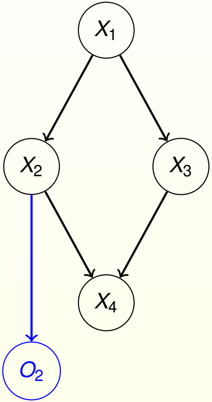

$\underline{K}(O_2|x_2)$ vacuous CS

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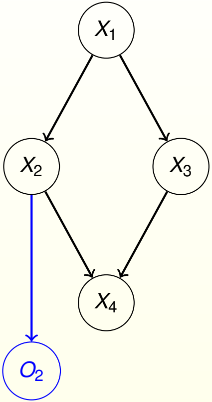

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$$= \underline{P}(x_4 | warm, O_2 = 1)$$

Modelling the observational process

- Each X as a *latent* variable
- For each X a *manifest* variable O_X modelling the observation $\Omega_O = \Omega_X \cup \{*\}$
- Conditional independence, given X between O and the other variables (or weaker conditions)
- Quantifying link between O and X (observational process)
- A CS $K(O|x)$ might a realistic model! (better than $P(O|X)$)
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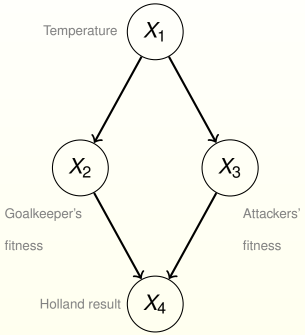

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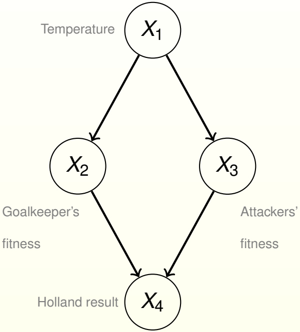

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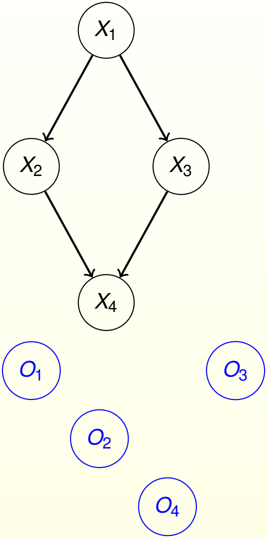

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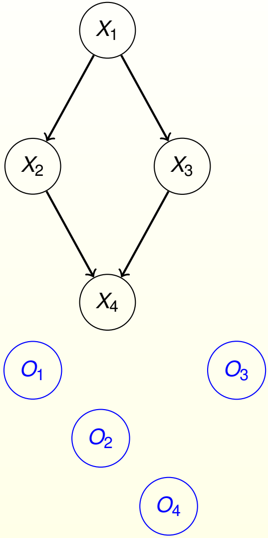

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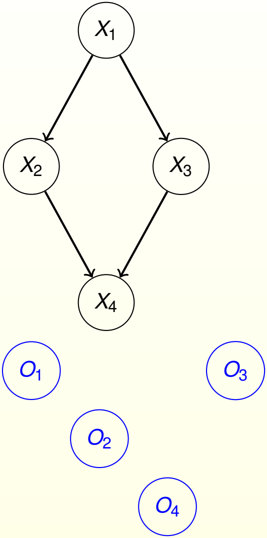

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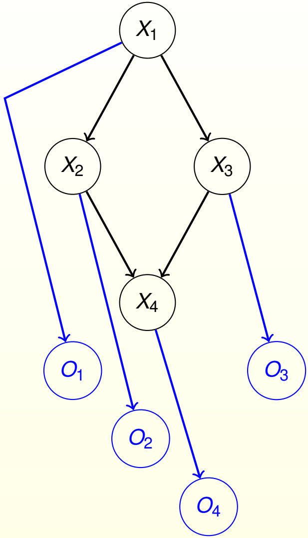

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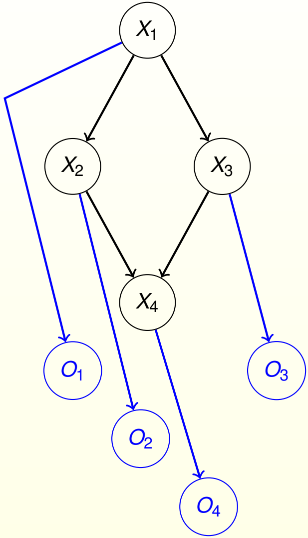

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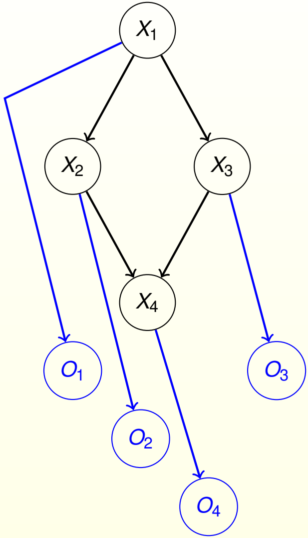

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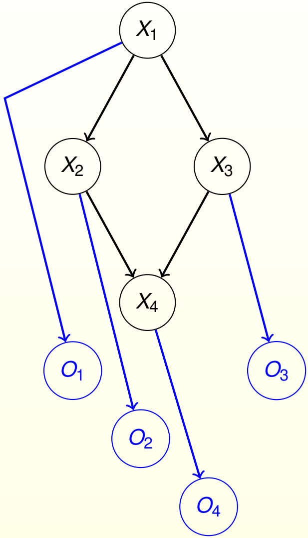

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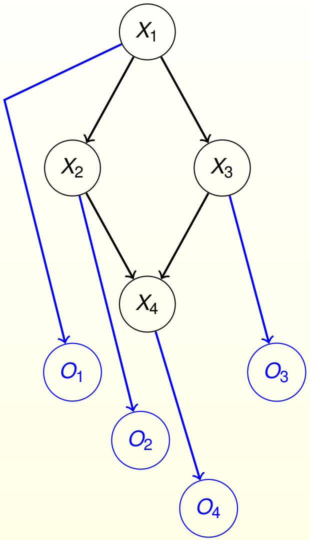

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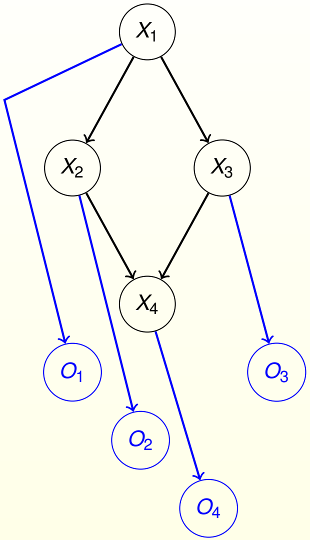

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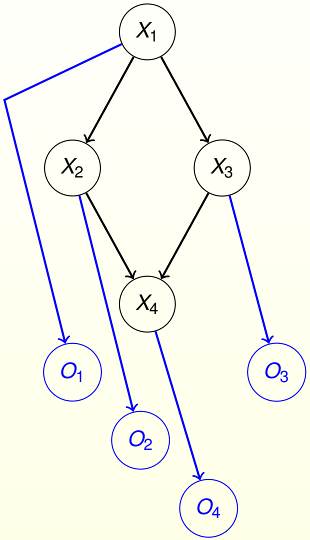

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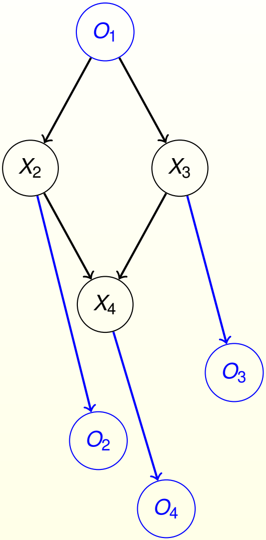

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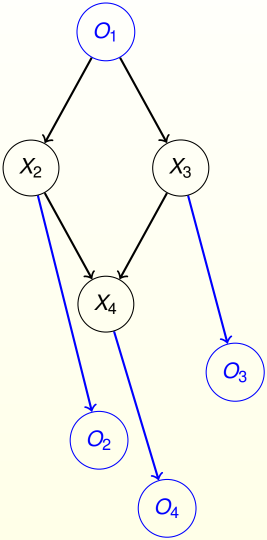

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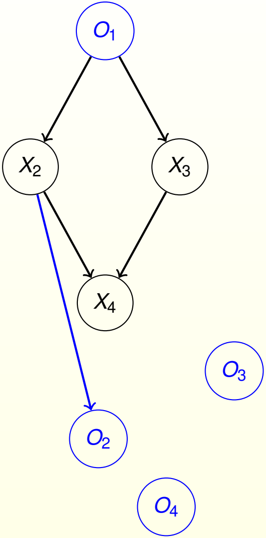

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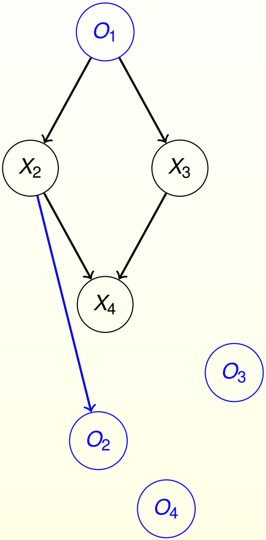

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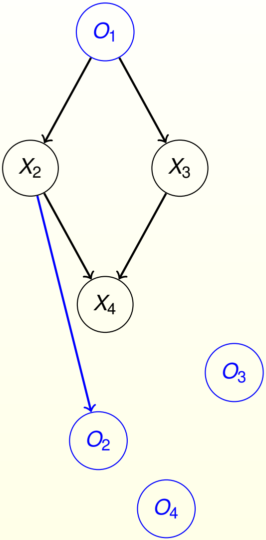

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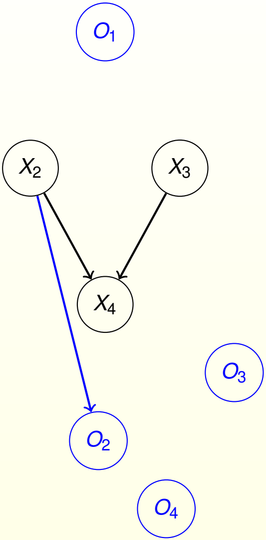

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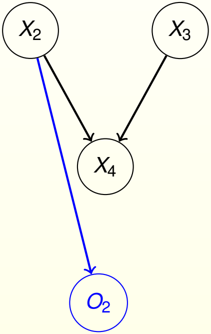

[Exe # 3] Is the ball in or out?

- B, with $\mathcal{B} = \{1, 0\}$, means the ball was in
- R, L_1, L_2 are the opinions/observation of the referee/linesmen
- A CN over these variables
- Given B, the three opinions are independent? Not really, the referee has an influence on the linesmen
- Compute bounds of
  $P(B = 1|R = 1, L_1 = 1, L_2 = 1)$
  $\in [.896, .962]$
  $P(B = 1|R = 0, L_1 = 1, L_2 = 1)$
  $P(B = 1|R = 1, L_1 = 0, L_2 = 0)$
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$P(B = 1) = .50$
$P(R = 1|B = 1) \in [.80, .90]$
$P(R = 1|B = 0) \in [.20, .30]$
$P(L_j = 1|B = 1, R = 1) \in [.90, .95]$
$P(L_j = 1|B = 1, R = 0) \in [.50, .60]$
$P(L_j = 1|B = 0, R = 1) \in [.40, .50]$
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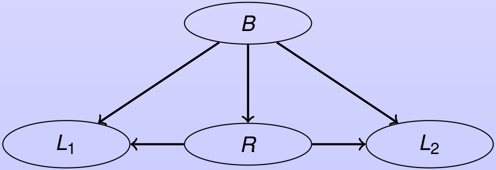


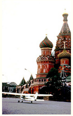
No-fly zones surveyed by the Air Force

- Around important potential targets (eg. WEF, dams, nuke plants)
- Twofold circle wraps the target
  - External no-fly zone (sensors)
  - Internal no-fly zone (anti-air units)
- An aircraft entering the zone (to be called **intruder**)
- Its presence, speed, height, and other features revealed by the sensors
- A team of military experts decides:
  - what the intruder intends to do (external zone / credal level)
  - what to do with the intruder (internal zone / pignistic level)

Identifying intruder's goal

- Four possible (exclusive, exhaustive) options for intruder's goal

renegade



provocateur

damaged

erroneous

- This identification is difficult
  - Sensors reliabilities are affected by geo/meteo conditions
  - **Information fusion** from several sensors

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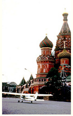
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Why credal networks?

- Why a **probabilistic** model?
  - No deterministic relations between the different variables
  - Pervasive uncertainty in the observations
- Why a **graphical** model?
  - Many independence relations among the different variables
- Why an **imprecise** (probabilistic) model?
  - Expert evaluations are mostly based on qualitative judgements
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- Intruder's goal and features as **categorical variables**
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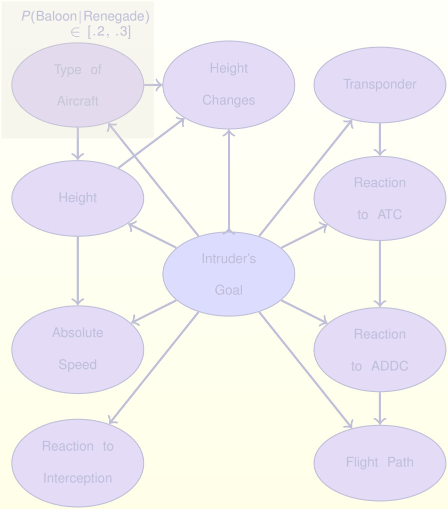
- A (small) **credal network**
- Complex observation process!



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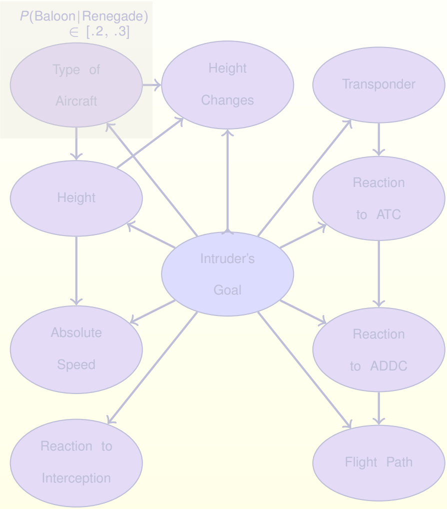

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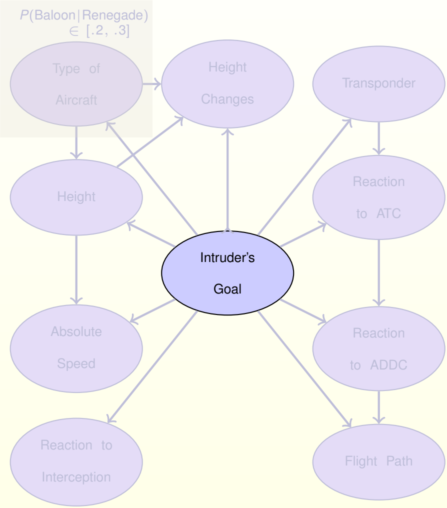

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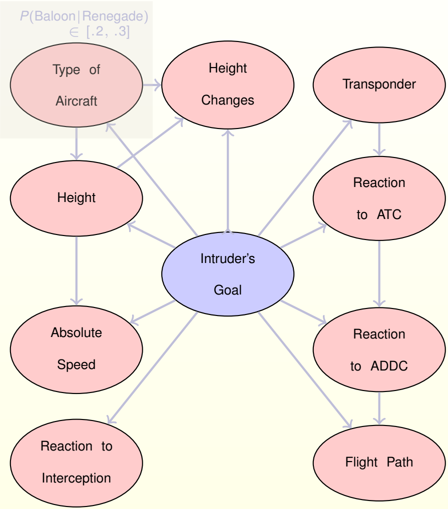

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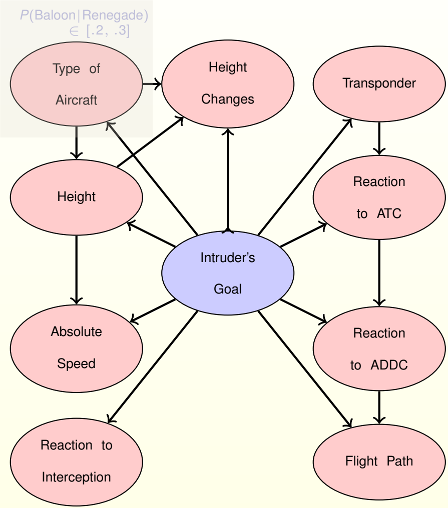

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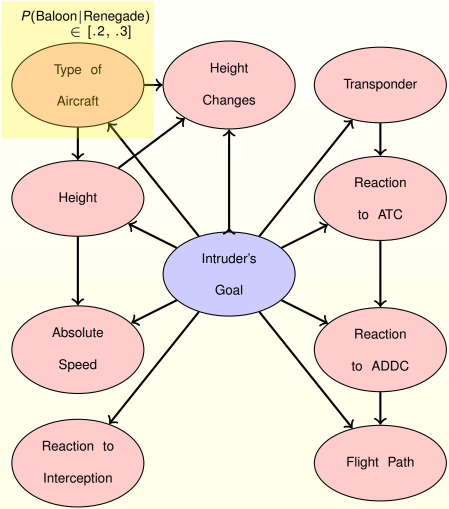

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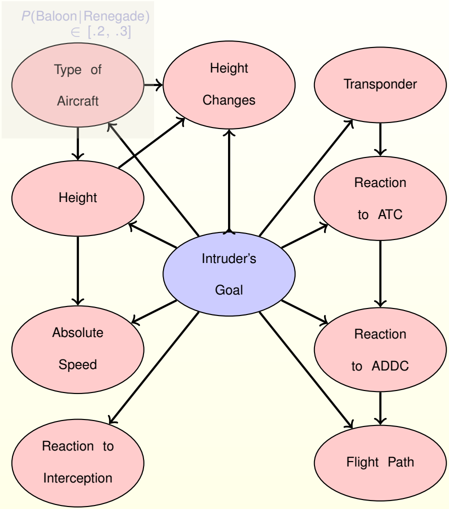

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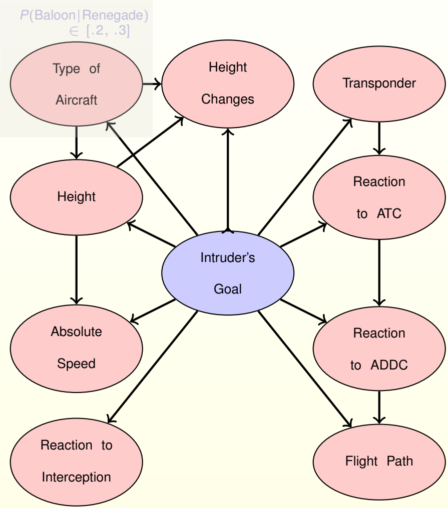

Observations Modeling and Fusion by Credal Nets

- Each sensor modeled by an auxiliary child of the (ideal) variable to be observed
- $P(\text{sensor}|\text{ideal})$ models sensor reliability

  (eg. identity matrix = perfectly reliable sensor)
- Many sensors? Many children!

  (conditional independence between sensors given the ideal)

Observations Modeling and Fusion by Credal Nets

- Each sensor modeled by an auxiliary child of the (ideal) variable to be observed
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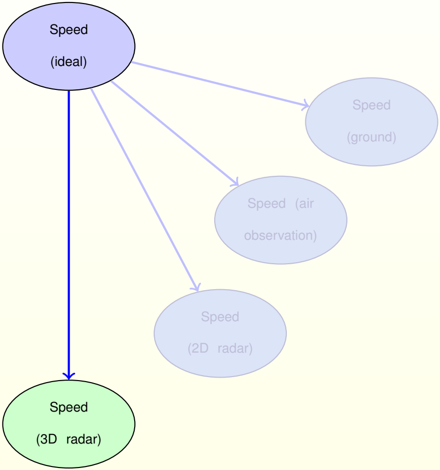

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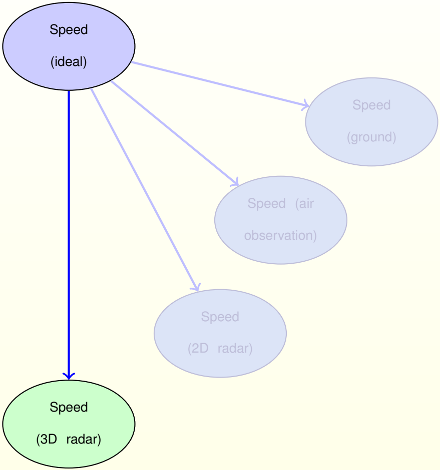

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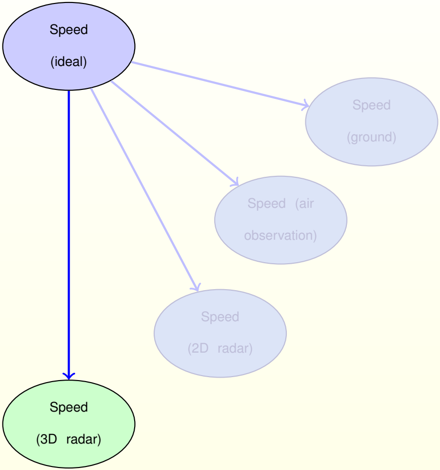

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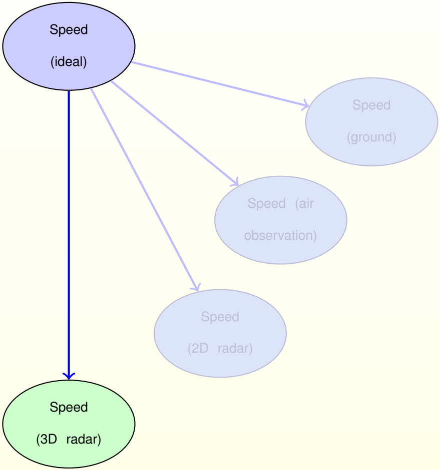

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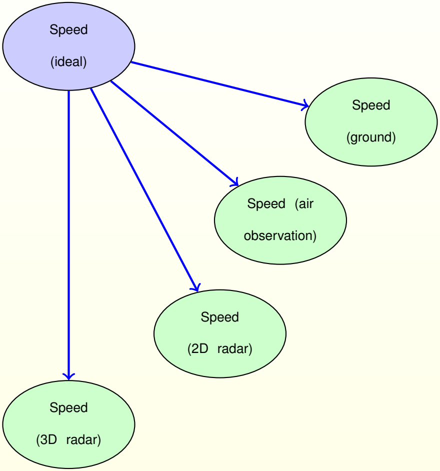

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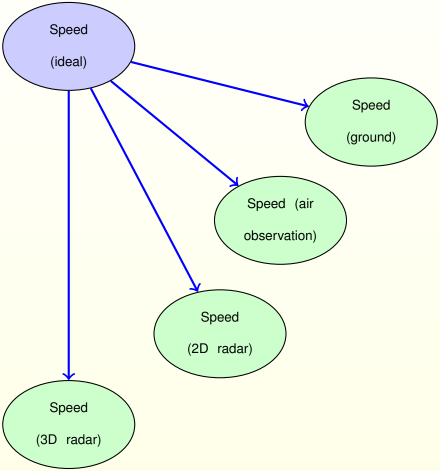

The whole network

- We conclude a huge multiply-connected credal network
- Approximate algorithm:
  1. Local specification [Antonucci and Zaffalon, PGM'06]
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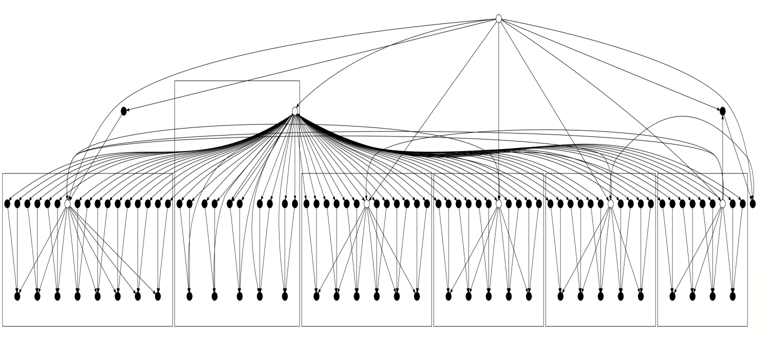

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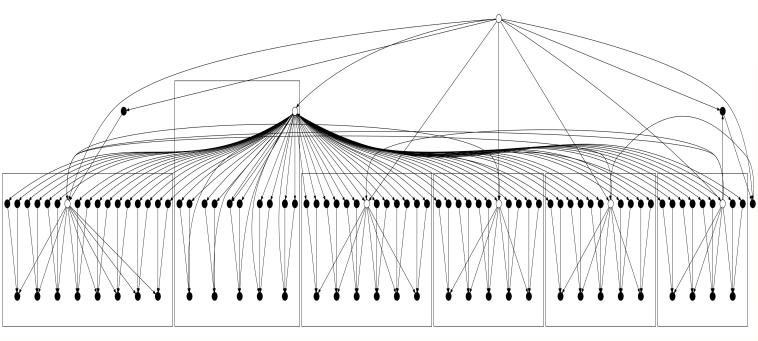

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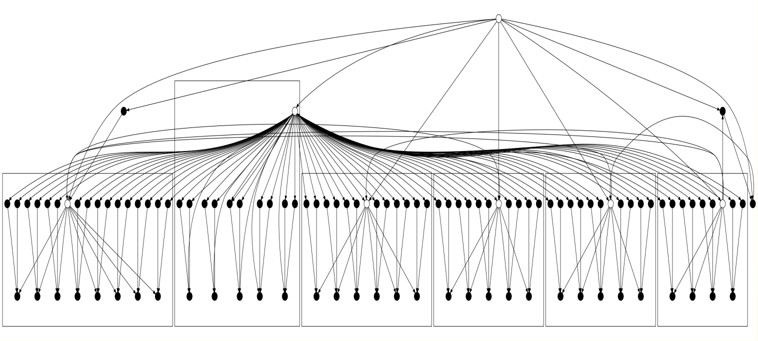

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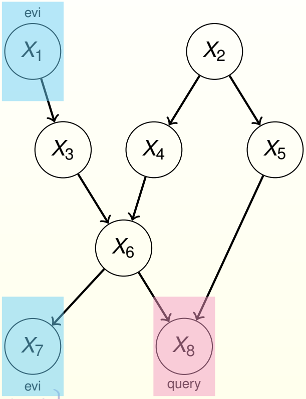

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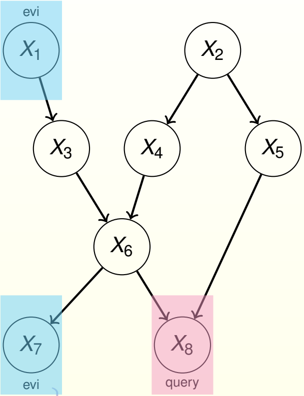

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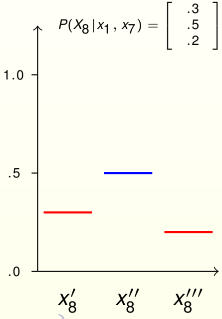

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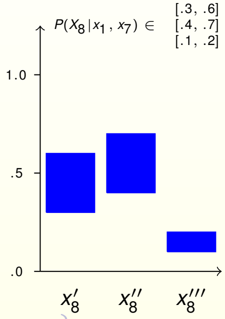

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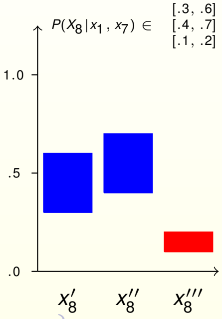

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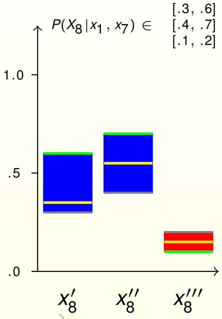

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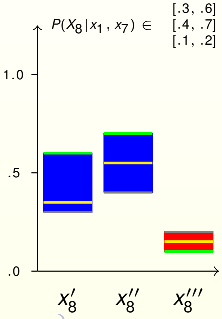

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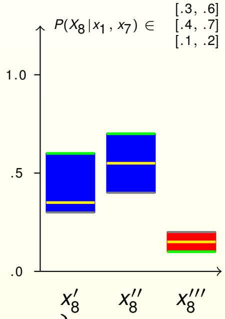

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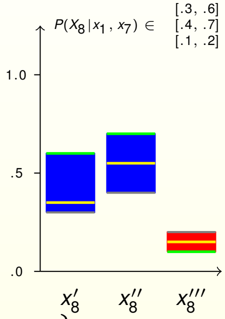

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- Simulating a dam in the Swiss Alps, with no interceptors, relatively good coverage for other sensors, discontinuous low clouds and daylight
- Sensors return:
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  - Type = helicopter / helicopter
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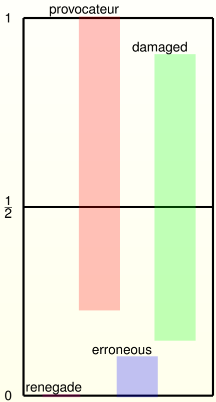

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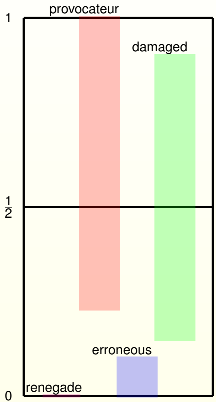

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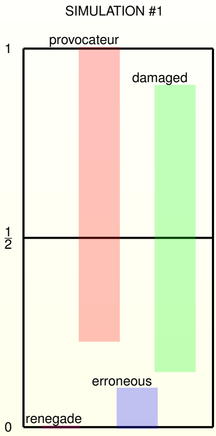

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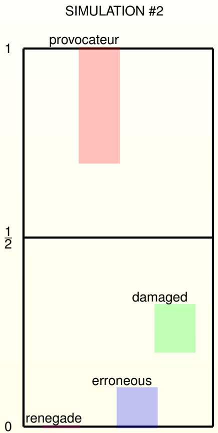

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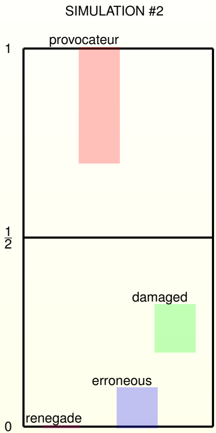
- We reject renegade and damaged, but indecision between provocateur and erroneous
- Assuming higher levels of reliability
- We conclude the aircraft is a provocateur!
- Same conclusion with maximality (no need of higher reliability)



Simulations (military application)

- Simulating a dam in the Swiss Alps, with no interceptors, relatively good coverage for other sensors, discontinuous low clouds and daylight
- Sensors return:
  - Height = very low / very low / very low / low
  - Type = helicopter / helicopter
  - Flight Path = U-path / U-path / U-path / U-path / U-path / missing
  - Height Changes = descent / descent / descent / descent / missing
  - Speed = slow / slow / slow / slow / slow
  - ADDC reaction = positive / positive / positive / positive / positive / positive
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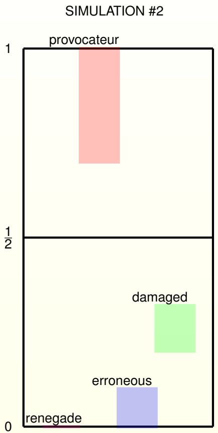

An application: debris flows risk assessment

- Debris flows are very destructive natural hazards
- Still partially understood
- Human expertize is still fundamental!
- An artificial expert system supporting human experts?

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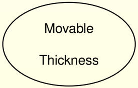
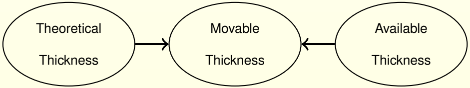
Building the causal network

Proxy indicator of the level of risk



Building the causal network

Triggering Factors



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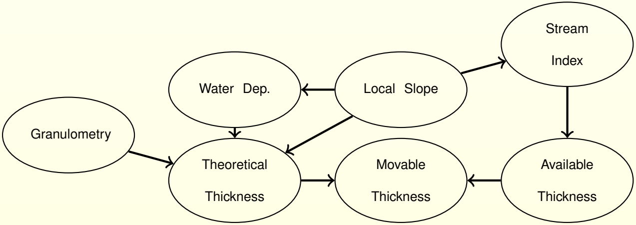

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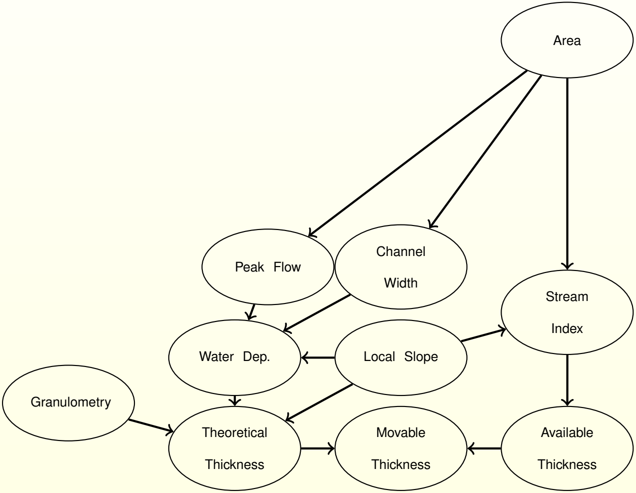

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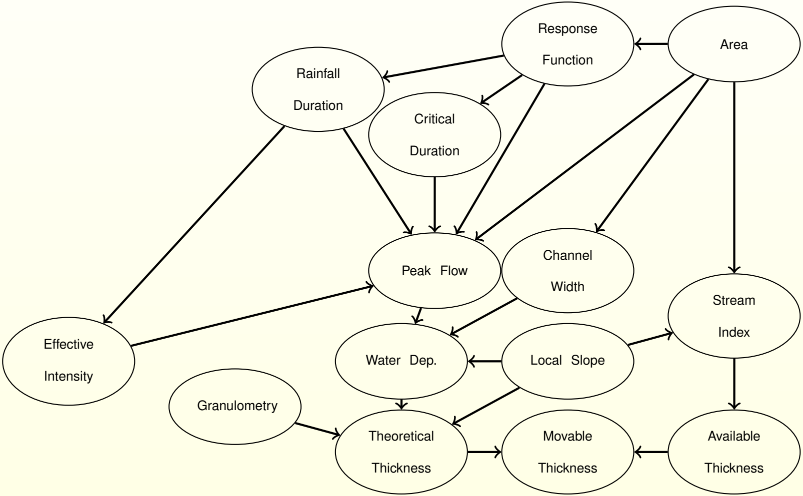

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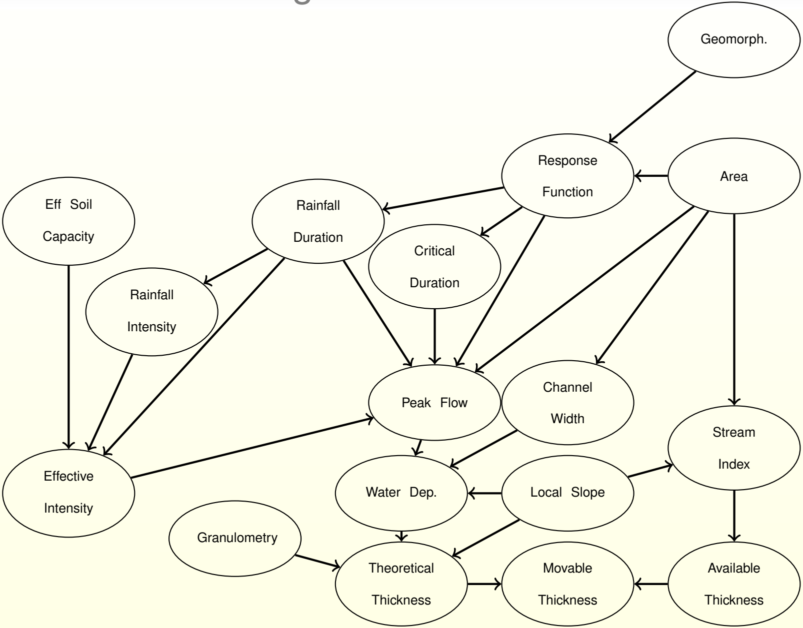

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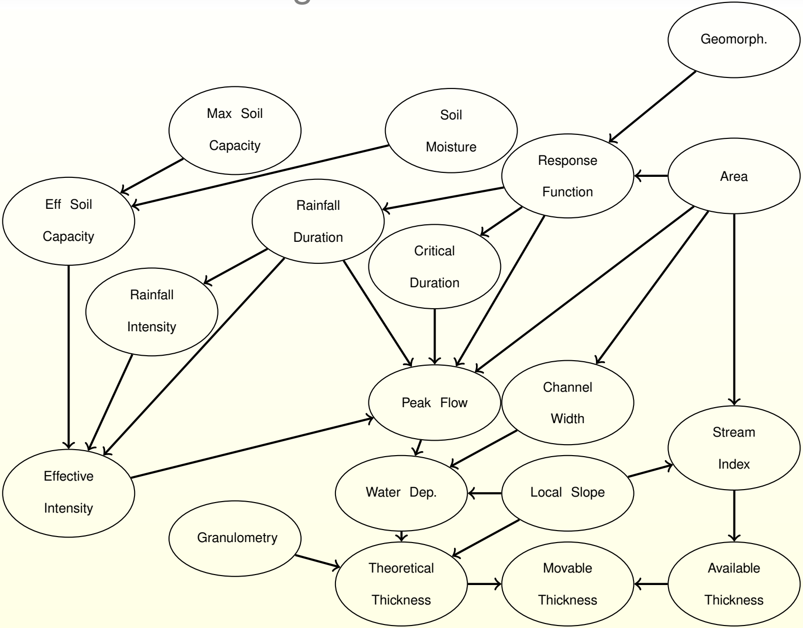

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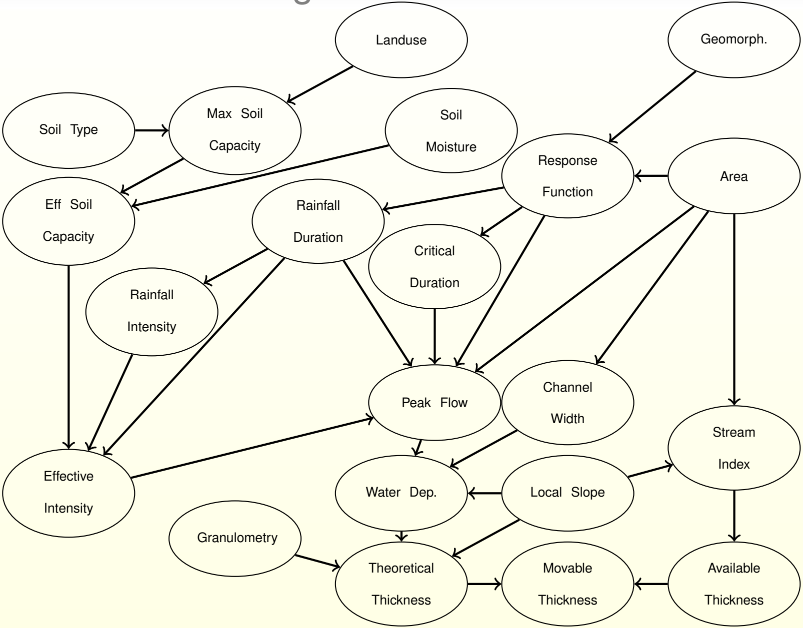

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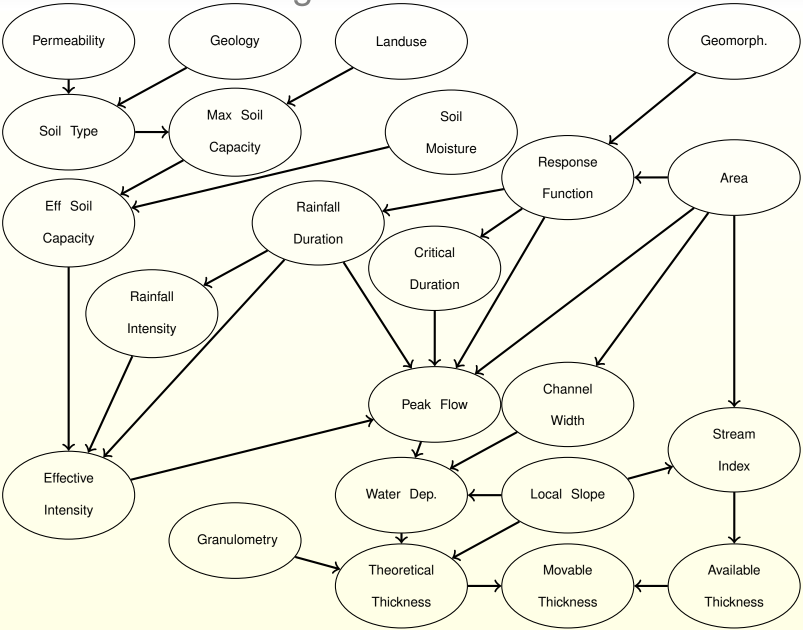
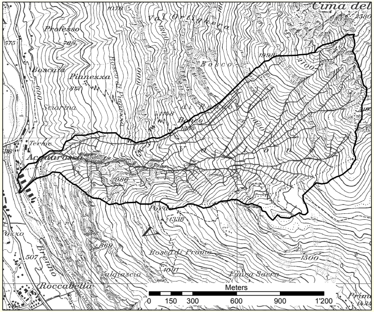


Debris flow hazard assessment by CNs

- Extensive simulations in a debris flow prone watershed Acquarossa Creek Basin (area 1.6 Km^2, length 3.1 Km)



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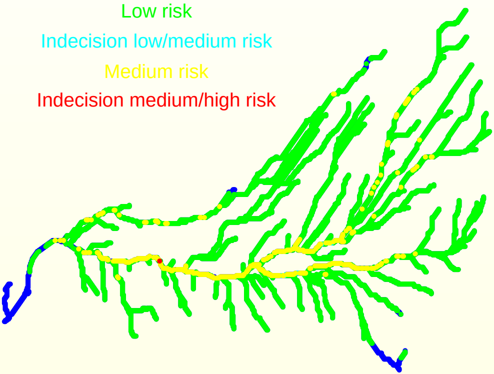

Inference based on message propagation

Bayesian nets

- Pearl's message propagation
  Efficient for polytrees
- Multiply connected BNs?
  Loopy belief propagation

Credal nets

- Only outer approximation for general polytrees
  (Tessem, 1992)
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- Exact for binary polytrees
  (2U, Zaffalon, 1998)
- Loopy version of 2U for binary multiply connected
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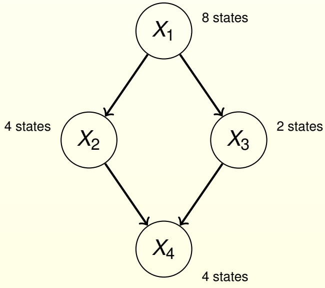
Updating non-binary CNs?

Binarizing non-binary credal nets

- State of a variable as a joint state of a number of "bits"
  $X = x \iff (\tilde{X}^1 = \tilde{x}^1) \wedge (\tilde{X}^2 = \tilde{x}^2) \wedge \ldots$
- For each arc between two variables, all the relative bits are linked, bits of the same variable are completely connected
- Local computations for the probabilities
- A "binarized" equivalent CN is obtained
- L2U can update it *(GL2U, Antonucci et al. 2010)*



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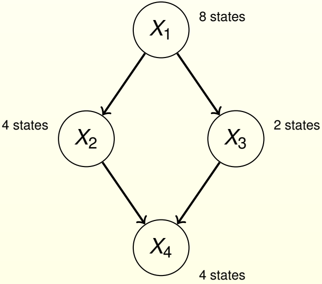

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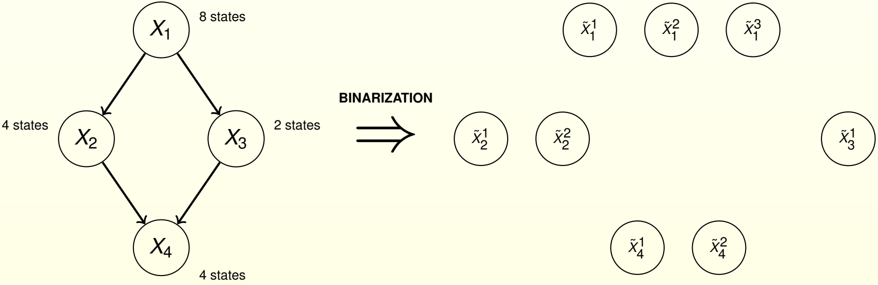

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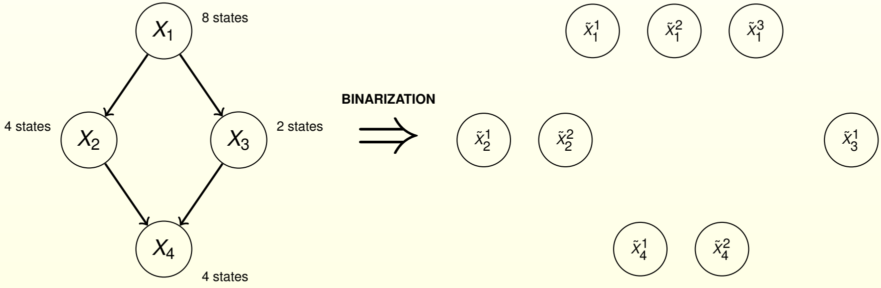

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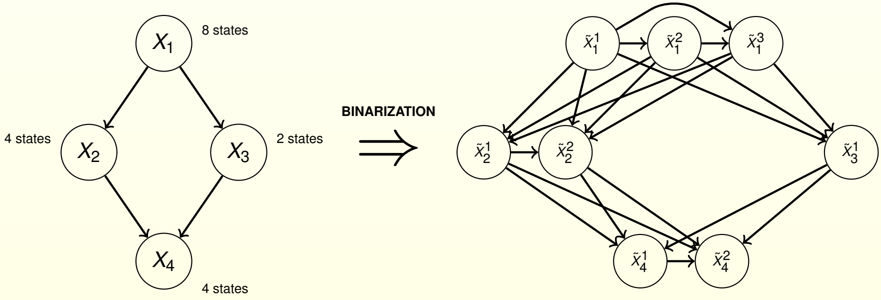

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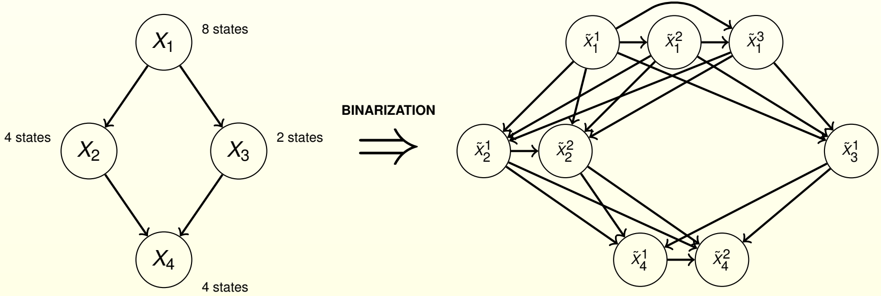

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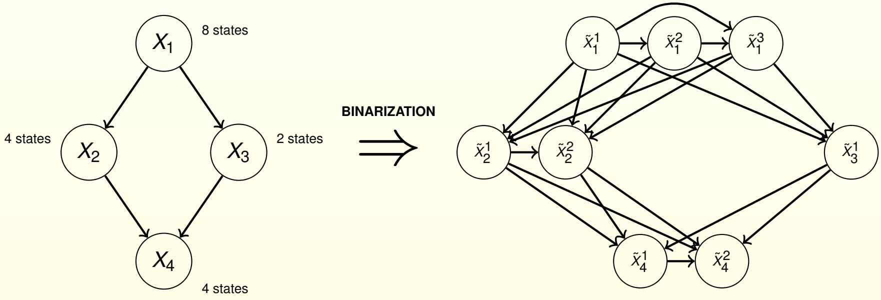

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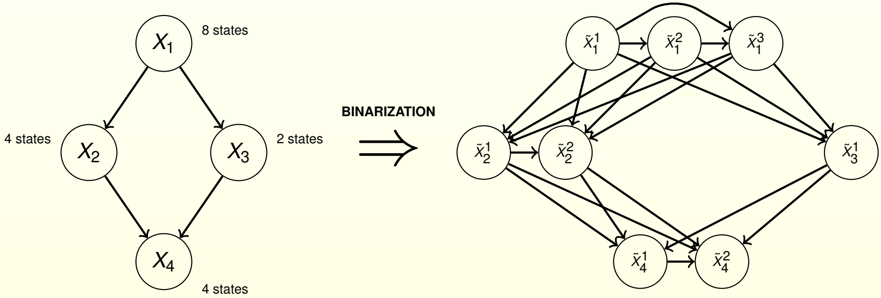

Exact inference: Variable elimination

BAYESIAN NETS

- Choose an ordering of the variables (query last)
- Create a *pool* of functions with all local distributions
- For each X:

  Insert all functions that contain X in a structure called *bucket* of X and remove them from the pool

  Multiply these functions and marginalize out X

  Insert the results in the pool

bucket elimination (Dechter, 1996)
fusion algorithm for valuation algebras (Shenoy & Kohlas, 1994)

Compute $P(X_4)$ with ordering X_1, X_2, X_3, X_4
Pool $\equiv \{P(X_1), p(X_2|X_1), p(X_3|X_1), p(X_4|X_2, X_3)\}$
Bucket(X_1): $\sum_{X_1} P(X_2|X_1)p(X_3|X_1)p(X_1) = p(X_2, X_3) \to$ pool
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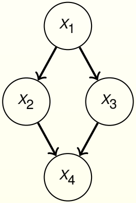
Bucket(X_4): just get $P(X_4)$ from the pool



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- Symbolic variable elimination
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- Updating $\equiv$ multilinear optimization
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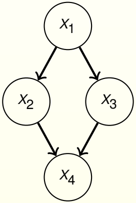

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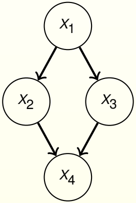

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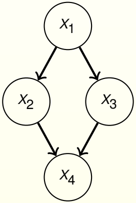
Bucket(X_4): just get $P(X_4)$ from the pool



CREDAL NETS

- Symbolic variable elimination
  multilinear constraints
- Updating $\equiv$ multilinear optimization
  (de Campos & Cozman, 2004)

Exact inference: Variable elimination

BAYESIAN NETS

- Choose an ordering of the variables (query last)
- Create a *pool* of functions with all local distributions
- For each X:

  Insert all functions that contain X in a structure called *bucket* of X and remove them from the pool

  Multiply these functions and marginalize out X

  Insert the results in the pool

bucket elimination *(Dechter, 1996)*
fusion algorithm for valuation algebras
(Shenoy & Kohlas, 1994)

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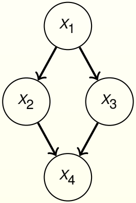

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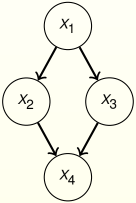

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- Inner approximation by iterative local search
  - Choose a BN consistent with the CN,
    vary parameters of a single node to improve the solution
    (da Rocha, Campos & Cozman, 2003)
- Outer approximation with *probability trees* *(Cano & Moral, 2002)*
- Integer linear programming *(de Campos & Cozman, 2007)*
- Branch and bound techniques on vertices
- Instead of propagating all the elements in the convex hull
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Other guys

Other IPGMs

- CNs with epistemic irrelevance (de Cooman) and epistemic independence (Cozman)
- Imprecise Markov Chains (Skulj)
- Hierarchical models (Cattaneo)
- Imprecise Markov decision processes (MDP) (Cozman)
- Qualitative probabilistic nets (Van der Gaag)
- Possibilistic networks (PGM with BFs)
- Imprecise decision Trees (Ekenberg, Jaffray)

Still to be formalized

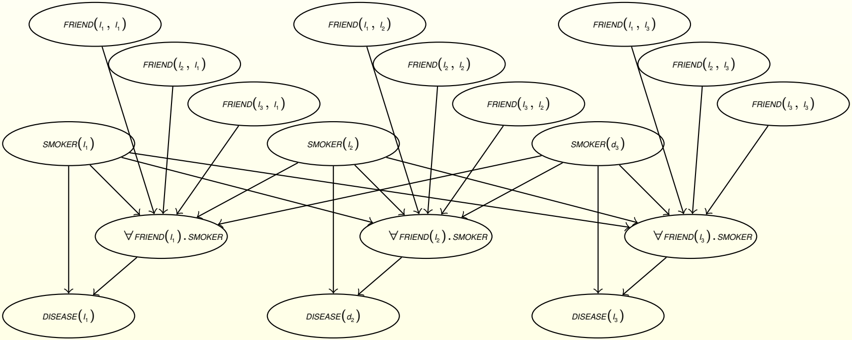
- Imprecise Markov random fields and iHMM
- Imprecise influence diagrams

Links with CNs

- Precise influence diagrams, MAP problems on BNs, . . .

$CR\mathcal{ALC}$ probabilistic logic with IPs *(Cozman, 2008)*

- Description logic with interval of probabilities
- N individuals $(I_1, \ldots, I_n)$,
  $P(smoker(I_i)) \in [.3, .5]$, $P(friend(I_j, I_i)) \in [.0, .5]$,
  $P(disease(I_i)|smoker(I_i), \forall friend(I_j, I_i).smoker(I_i)) = \ldots$
- $\underline{P}(disease)$? Inference $\equiv$ updating of a (large) binary CN
- In a sens symbolic (or OO) CNs



Future directions for CNs

- Inference algorithms
  - Inference based on Pareto set (de Campos)
  - Gibb's sampling
  - Joint tree
- Learning CNs from data
  - Structural learning (next talk)
  - Imprecise EM
- More "bridges" with BNs world
- Continuous variables (Benavoli)
- Undirected Models (random Markov fields with imprecision)
- Applications, applications, applications, applications, applications