Imprecise Markov chains, exercises
4th SIPTA Summer School

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Durham, UK, 1-6 September 2010

1. We are listening to music on the radio. There are there stations with blocks of music and blocks of commercials. When commercials start on the station that we are listening at the moment we switch to another station. We have three possibilities:

   (i) with probability 0.2 there are no commercials on other two stations, in which case we select the new station with equal probability;

   (ii) with probability 0.1 respectively there are commercials on exactly one of the remaining stations but not on the other, in which case we switch to the station without commercials;

   (iii) with probability 0.6 there are commercials on all three stations, in which case we select one of them with equal probability and wait until the end of the commercial block.

   Denote stations with x_1, x_2, x_3 and consider the Markov chain formed by the sequence of stations listened to.

   (a) Find the transition matrix for the Markov chain.

   (b) What is the probability that we will be listening to the station x_3 after 3 switchings, if we are listening to x_1 at the moment?

   (c) Does the chain converge? What is the limit probability distribution?

   (d) Calculate the coefficient of ergodicity.

   (e) Consider the probability that we will be listening to the station x_1 after 5 switchings. How much can this probability differ from the long term probability?

2. The following lower and upper transition matrices are given for an imprecise Markov chain with states x_1, x_2, x_3:

$$\underline{P} = \begin{bmatrix} 0.2 & 0.1 & 0.3 \\ 0.1 & 0.4 & 0.2 \\ 0.6 & 0.1 & 0.1 \end{bmatrix} \text{ and } \overline{P} = \begin{bmatrix} 0.6 & 0.7 & 0.5 \\ 0.5 & 0.7 & 0.4 \\ 0.9 & 0.2 & 0.4 \end{bmatrix}$$

(a) Are the intervals coherent, i.e. are all lower and upper bounds reachable? If not, find the corresponding coherent bounds. In sequel assume the coherent bounds.

(b) Find the lower probability of being in x_2 after two steps, if currently we are in x_1.

(c) Write the upper transition matrix, i.e. the matrix containing upper 1-step probabilities for all non-trivial subsets in the lexicographic order.

(d) Calculate the uniform and the weak coefficient of ergodicity.

3. Consider Exercise 1 again.

   (a) Suppose that in case (i) the new station is selected with an unknown (vacuous) probability. What is the lower and upper transition matrix then?

   (b) Suppose that both in case (i) and (iii) the new station is selected with an unknown (vacuous) probability. Calculate the lower and the upper transition matrix.

   (c) For both cases find the lower and upper probability that after two switchings we will be listening to the same station as at the beginning.

4. Suppose that three levels of education are possible: *primary, secondary* and *higher*. Consider the education of women. We have the following judgments about daughters of women with primary, secondary and higher education:

   primary at least 20% of their daughters achieve higher education, but no more than 40%; at least 40% achieve only primary education;

   secondary at least 30% of their daughters achieve higher education and at least 80% at least secondary education;

   higher at least 90% of their daughters achieve at least secondary education and at least 50% achieve higher education.

   (a) Write the lower and upper transition matrix.

   (b) What is the upper probability that a granddaughter of a woman with secondary education achieves only primary education?

5. [Theoretical exercise.] Let P_1 and P_2 be (precise) transition matrices of two Markov chains. Consider the matrix $P = \alpha P_1 + (1 - \alpha)P_2$, where $0 < \alpha < 1$. Prove that a Markov chain with the transition matrix P is convergent if at least one of the two Markov chains are convergent. Hint: use coefficients of ergodicity.

The calculations can be done with the `Matlab/Octave` functions. Edit the files `markov_chain.m` or `markov_chain_pri.m` to specify parameters and run with `Matlab/Octave`.

The input in the file `markov_chain.m` is a lower transition matrix. It can be calculated from the upper transition matrix by using identity: $L(A) = 1 - U(A^c)$.