Nonparametric predictive inference for system reliability with redundancy allocation

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[†]Pauline Coolen-Schrijner died on 23 April 2008; this paper is dedicated to her memory

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Abstract: This paper presents lower and upper probabilities for the reliability of k-out-of-m systems, which include series and parallel systems, and of series systems with independent k_i-out-of-m_i subsystems, for which optimal redundancy allocation is also presented in case of zero-failure testing. First, attention is restricted to k-out-of-m systems with exchangeable components. The lower and upper probabilities for successful functioning of the system are based on the nonparametric predictive inferential (NPI) approach for Bernoulli data. In this approach, it is assumed that test data are available on the components, and that the future components to be used in the system are exchangeable with these. Thereafter, systems are considered that consist of a series of independent subsystems, with subsystem i a k_i-out-of-m_i system consisting of exchangeable components. For such systems, an algorithm for optimal redundancy allocation after zero-failure testing is presented. A particularly attractive feature of NPI in reliability, with lower and upper probabilities, is that data containing zero failures can be dealt with in an attractive manner.

Keywords: k-out-of-m systems, lower and upper probabilities, nonparametric predictive inference, redundancy allocation, series-parallel systems, system reliability, zero-failure testing

1 INTRODUCTION

During the last decade, imprecise probabilistic methods in reliability have received increasing attention; concise overviews are presented by Coolen [**1**] and by Coolen and Utkin [**2**], while a detailed introduction and overview is given by Utkin and Coolen [**3**]. These methods are based on generalized uncertainty quantifications via lower and upper probabilities, also known as the theory of imprecise probability [**4**] or interval probability [**5**, **6**]. During this period, Coolen and several collaborators (see reference [**7**] for a brief overview) have developed a novel statistical theory entitled nonparametric predictive inference (NPI), with an early overview of possible applications of NPI in reliability presented by Coolen *et al.* [**8**]. Recently presented applications of NPI to reliability problems include reliability demonstration for failure-free periods [**9**], (opportunity-based) age replacement [**10**, **11**], comparison of success–failure data [**12**], probabilistic safety assessment in case of zero failures [**13**], and prediction of not yet observed failure modes [**14**].

A traditional problem considered in reliability theory is the assessment of system reliability [**15**], where k-out-of-m systems, also called 'voting systems', have received particular attention. These are systems that consist of m exchangeable components (often the confusing term 'identical components' is used), such that the system functions if and only if at least k of its components function, with series systems ($k = m$) and parallel systems ($k = 1$) as special cases. Utkin [**16**] considered such systems for situations with incomplete information, using imprecise probability to quantify uncertainty. In this present paper, Coolen's NPI method for Bernoulli data is applied to k-out-of-m systems, where it is assumed that inferences on system reliability are based on information from tests on n components, which are exchangeable with the components in the system considered. Throughout,

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only situations are considered where components, and therefore also the system, either function or not when called upon, so failure behaviour over time is not considered. Generally, it is assumed that a test of n exchangeable components reveals that s of them function and $n-s$ fail, and NPI is used to derive the lower and upper probabilities for the event that the k-out-of-m system, made up of components exchangeable with those n tested, functions. For the extension to series of independent k_i-out-of-m_i subsystems, in sections 5 and 6, the same scenario is assumed, with the components per subsystem exchangeable, and with tests of n_i components exchangeable with those of subsystem i revealing s_i failures.

Section 2 presents a brief overview of NPI, and particularly, of NPI for Bernoulli random quantities, as used later in this paper. Section 3 presents the main results on NPI for k-out-of-m systems, and these results are illustrated and discussed via examples in section 4. Section 5 extends this approach to systems that are series of independent subsystems, with each subsystem a k_i-out-of-m_i system with exchangeable components. For such systems, the question of how to allocate additional components to the subsystems to increase redundancy, and to obtain a maximum increase in system reliability, has received much attention in the literature, and tends to be a complex problem. In the NPI setting, however, it turns out that this problem can easily be solved by applying an algorithm that is presented in section 6. Section 7 provides some concluding remarks, including a brief outline of related research challenges. The material in section 3, with the illustrative examples in section 4, was presented at the Mathematical Methods in Reliability Conference in 2007 (Glasgow, UK), and a short article with those results has appeared in a book with special invited papers from that conference [**17**].

2 NONPARAMETRIC PREDICTIVE INFERENCE

In this section, results from Coolen [**18**] on NPI for Bernoulli random quantities are presented, which are based on a representation of Bernoulli data as outcomes of an experiment similar to that used by Bayes [**19**], with Hill's assumption $A_{(n)}$ [**20**, **21**] used to derive direct predictive probabilities [**22**, **23**] for future observations using available data. For detailed justifications of these results the reader is referred to reference [**18**]. The lower and upper probabilities presented by Coolen [**18**] fit in the framework of NPI [**7**, **24**], hence, they will also be called NPI (-based) lower and upper probabilities in this paper, and they have strong internal consistency properties in the theory of interval probability [**6**, **24**] as can be proven similarly to the proofs of such consistency properties for the more general situation of multinomial data [**25**]. Due to the use of $A_{(n)}$ in deriving these lower and upper probabilities, they fit in a frequentist framework of statistics, but can also be interpreted from a Bayesian perspective [**21**, **26**]. As they are conditional lower and upper probabilities, which are introduced without reference to probabilities for the unconditional events, they can be interpreted in a way similar to Dempster's 'direct probabilities' [**22**]. NPI is also 'exactly calibrated' in the frequentist sense of Lawless and Fredette [**27**]. For further discussion of such inferences, see Augustin and Coolen [**24**] and Coolen [**7**]. The NPI approach for Bernoulli random quantities [**18**] has been used for several other applications; for example, for multiple comparisons of proportions [**28**], where also particular attention has been paid to reliability data with few or zero failures [**12**].

Suppose that there is a sequence of $n+m$ exchangeable Bernoulli trials, each with 'success' and 'failure' as possible outcomes, and data consisting of s successes in n trials. Let Y_1^n denote the random number of successes in trials 1 to n, then a sufficient representation of the data for the inferences considered is $Y_1^n = s$, due to the assumed exchangeability of all trials. Let Y_{n+1}^{n+m} denote the random number of successes in trials $n+1$ to $n+m$. Let $R_t = \{r_1, \ldots, r_t\}$, with $1 \leqslant t \leqslant m+1$ and $0 \leqslant r_1 < r_2 < \ldots < r_t \leqslant m$, and, for ease of notation, define $\binom{s+r_0}{s} = 0$. Then, the NPI-based upper probability [**18**] for the event $Y_{n+1}^{n+m} \in R_t$, given data $Y_1^n = s$, for $s \in \{0, \ldots, n\}$, is

$$\overline{P}(Y_{n+1}^{n+m} \in R_t | Y_1^n = s) = \binom{n+m}{n}^{-1} \times \sum_{j=1}^{t} \left[\binom{s+r_j}{s} - \binom{s+r_{j-1}}{s} \right] \binom{n-s+m-r_j}{n-s}$$

The corresponding NPI lower probability is derived via the conjugacy property [**18**]

$$\underline{P}(Y_{n+1}^{n+m} \in R_t | Y_1^n = s) = 1 - \overline{P}(Y_{n+1}^{n+m} \in R_t^c | Y_1^n = s)$$

where $R_t^c = \{0, 1, \ldots, m\} \backslash R_t$. This conjugacy property between these upper and lower probabilities is justified in reference [**18**], and agrees with the fact that they are F-probabilities in Weichselberger's theory of interval probability [**5**, **6**]. These lower and upper probabilities also have attractive properties beyond internal consistency, as the interval created by the lower and upper probability for an event A always contains the precise empirical probability for A as based on the observed data, and the lower (upper) probability increases (decreases) as a function of n, for constant s/n [**7**].

3 NPI FOR k-OUT-OF-m SYSTEMS

In this section, the focus is on the lower and upper probabilities for the event $Y_{n+1}^{n+m} \geqslant k$ (for $m \geqslant 1$) given

the test results, as this event corresponds to successful functioning of a k-out-of-m system following n tests of components that are exchangeable with the m components in the system considered. Given data consisting of s successes from n components tested, the NPI lower and upper probabilities for the event that the k-out-of-m system functions successfully are also denoted by $\underline{P}(m:k|n,s)$ and $\overline{P}(m:k|n,s)$ respectively. From the upper probability for $Y_{n+1}^{n+m} \in R_t$ given in section 2 [**18**], the NPI upper probability $\overline{P}(m:k|n,s)$ easily follows. For $k \in \{1, 2, \ldots, m\}$ and $0 < s < n$, this NPI upper probability equals

$$\begin{aligned}\overline{P}(m:k|n,s) &= \overline{P}(Y_{n+1}^{n+m} \geqslant k | Y_1^n = s) \\ &= \binom{n+m}{n}^{-1} \left[\binom{s+k}{s} \binom{n-s+m-k}{n-s} \right. \\ &\quad \left. + \sum_{l=k+1}^{m} \binom{s+l-1}{s-1} \binom{n-s+m-l}{n-s} \right] \end{aligned} \quad (1)$$

and the NPI lower probability follows via the conjugacy property

$$\begin{aligned}\underline{P}(m:k|n,s) &= \underline{P}(Y_{n+1}^{n+m} \geqslant k | Y_1^n = s) \\ &= 1 - \overline{P}(Y_{n+1}^{n+m} \leqslant k-1 | Y_1^n = s) \\ &= 1 - \binom{n+m}{n}^{-1} \left[\sum_{l=0}^{k-1} \binom{s+l-1}{s-1} \right. \\ &\quad \left. \times \binom{n-s+m-l}{n-s} \right] \end{aligned} \quad (2)$$

For $m = 1$, so considering a system consisting of just a single component, the NPI upper and lower probabilities for the event that the system functions successfully are

$$\overline{P}(1:1|n,s) = \overline{P}(Y_{n+1}^{n+1} = 1 | Y_1^n = s) = \frac{s+1}{n+1}$$

and

$$\underline{P}(1:1|n,s) = \underline{P}(Y_{n+1}^{n+1} = 1 | Y_1^n = s) = \frac{s}{n+1}$$

If the observed data are all successes, so $s = n$, or all failures, so $s = 0$, then the NPI upper probabilities are, for all $k \in \{1, \ldots, m\}$

$$\begin{aligned}\overline{P}(m:k|n,n) &= \overline{P}(Y_{n+1}^{n+m} \geqslant k | Y_1^n = n) = 1 \\ \overline{P}(m:k|n,0) &= \overline{P}(Y_{n+1}^{n+m} \geqslant k | Y_1^n = 0) \\ &= \binom{n+m-k}{n} \binom{n+m}{n}^{-1} \end{aligned}$$

and the NPI lower probabilities are, for all $k \in \{1, \ldots, m\}$

$$\begin{aligned}\underline{P}(m:k|n,n) &= \underline{P}(Y_{n+1}^{n+m} \geqslant k | Y_1^n = n) \\ &= 1 - \binom{n+k-1}{n} \binom{n+m}{n}^{-1} \\ \underline{P}(m:k|n,0) &= \underline{P}(Y_{n+1}^{n+m} \geqslant k | Y_1^n = 0) = 0 \end{aligned}$$

For the two extreme cases of series and parallel systems, with $k = m$ and $k = 1$, respectively, the NPI upper and lower probabilities can be substantially simplified to give the expressions below, which actually provide insight into the NPI approach for such systems. Representing corresponding lower and upper probabilities for an event A by $(\underline{P}, \overline{P})(A)$, the general results above are, for series systems

$$\begin{aligned}(\underline{P}, \overline{P})(m:m|n,0) &= \left(0, \prod_{j=1}^{m} \frac{j}{n+j}\right) \\ (\underline{P}, \overline{P})(m:m|n,s) &= \left(\prod_{j=1}^{m} \frac{s-1+j}{n+j}, \prod_{j=1}^{m} \frac{s+j}{n+j}\right) \\ &\quad \text{for } 0 < s < n \\ (\underline{P}, \overline{P})(m:m|n,n) &= \left(\frac{n}{n+m}, 1\right) \end{aligned}$$

and for parallel systems

$$\begin{aligned}(\underline{P}, \overline{P})(m:1|n,0) &= \left(0, \frac{m}{n+m}\right) \\ (\underline{P}, \overline{P})(m:1|n,s) &= \left(1 - \prod_{j=1}^{m} \frac{n-s+j}{n+j}, \right. \\ &\quad \left. 1 - \prod_{j=1}^{m} \frac{n-s-1+j}{n+j}\right) \quad \text{for } 0 < s < n \\ (\underline{P}, \overline{P})(m:1|n,n) &= \left(1 - \prod_{j=1}^{m} \frac{j}{n+j}, 1\right) \end{aligned}$$

In these expressions, a term $\binom{n+m}{n}^{-1}$ for example, is written as $\prod_{j=1}^{m} j/(n+j)$, as the latter expression provides more insight. For example, it is the value of the NPI upper probability for successful functioning of a series system when all the n components tested failed, $\overline{P}(m:m|n,0)$. In this situation, when the m components of the system are considered in sequence, then the NPI upper probability for the first to function, given the data, is $1/(n+1)$. If this first component in the system were to function, then the data of the n tested components would be used together with this functioning first component, to infer that the NPI upper probability for the event that the second component in the system functions, given that the first functions, is equal to $2/(n+2)$, and so on for all m components in sequence, with, for each one, both the information from the test and from the previous components of the system in this sequence used. It is easily seen that this leads to $\overline{P}(m:m|n,0) = \prod_{j=1}^{m} j/(n+j)$. The other expressions for these series and parallel systems can be derived and interpreted via similar direct considerations.

An important advantage of the use of lower and upper probabilities in statistical inference occurs in situations with the observations either all successes or all failures, as inferences based on precise probabilistic methods are typically not in agreement with empirical probabilities in such cases. For example,

after observing zero failures in tests of n components, it might be expected that a future component has a small probability of failure, but assigning the value 0 to this probability might not be considered to be appropriate for any finite value of n. However, observing zero failures in n components does not exclude the possibility that failures could never happen. An attractive, albeit informal, manner in which to interpret lower and upper probabilities is to regard the lower probability $\underline{P}(A)$ as quantifying the evidence in favour of event A, and the upper probability $\overline{P}(A)$ as quantifying the evidence against event A (hence, in favour of A^c, in agreement with the conjugacy property). From this perspective, when considering a system consisting of only a single component, the NPI lower and upper probabilities for successful functioning of this one future component, given zero failures in n components tested (so $s=n$), which are equal to $(\underline{P},\overline{P})(1:1|n,n)=(n/(n+1),1)$, are attractive, as the upper probability reflects that there is no evidence from the test data against successful functioning of the future component, whereas the lower probability provides a natural cautious inference that can be used in quantitative risk assessment. As such, the results in this paper can be used in zero-failure reliability demonstrations from NPI perspective, generalizing the results presented by Coolen and Coolen-Schrijner [**9**]. For example, it is possible to consider decisions on levels of redundancy to build into the system (e.g. the value of m if k is determined by the system requirements), if possible, in order to reduce the number of zero-failure component tests required to demonstrate reliability at a chosen level. This is briefly illustrated in Example 3 in section 4, and discussed in more detail in section 6 for more complex systems. The results in this paper can also be used in a straightforward manner to take into account the costs of components and of testing, together with practical constraints on test budget and time, in line with the Bayesian approach to such problems [**29**, **30**]. This is also briefly discussed in section 6, but it is mostly left as an interesting and important topic for future research.

For the special cases with $m=1$, $k=1$, or $k=m$, for which the NPI lower and upper probabilities for successful system functioning given s successes in n component tests are given above, it is easily seen that the following result holds, for $0 \leqslant s \leqslant n$

$$\overline{P}(m:k|n,s)=\underline{P}(m:k|n,s+1) \tag{3}$$

The result of equation (3) actually holds generally for the NPI lower and upper probabilities for all k-out-of-m systems as considered in this paper. A direct proof, using the expressions (1) and (2), is an exercise in combinatorial analysis. However, this result follows immediately from detailed consideration of the underlying representation assumed for Bernoulli random quantities in the NPI method by Coolen [**18**] that is used here. In this paper, this detailed justification for the equality (3) is not provided, but the examples in section 4 will, of course, illustrate this interesting property of these inferences. A second way to quickly justify the equality (3) is based on the fact that all NPI results for k-out-of-m systems presented in this section can also be derived within the Bayesian framework, which will be briefly discussed in section 7. The result (3) can obviously be used to reduce computational effort, if upper and lower probabilities are required for all possible values of s. The elegance of this equality should be emphasized, as it implies that the intervals created by corresponding lower and upper probabilities of successful system functioning, for $s=0,1,\ldots,n$, form a partition of the interval [0,1].

4 EXAMPLES FOR k-OUT-OF-m SYSTEMS

In this section, numerical examples are presented to illustrate the results from section 3, and some related issues are discussed.

Example 1

Consider a series system with ten exchangeable components (so, $k=m=10$), and the only information available is the result of a test of two components, also exchangeable with the ten to be used in the system. For the three possible values of the number of successes in the test, $s=0$, 1, 2, the NPI lower and upper probabilities for successful functioning of the system are

$$(\underline{P},\overline{P})(10:10|2,0)=\left(0,\frac{1}{66}\right)$$

$$(\underline{P},\overline{P})(10:10|2,1)=\left(\frac{1}{66},\frac{1}{6}\right)$$

$$(\underline{P},\overline{P})(10:10|2,2)=\left(\frac{1}{6},1\right)$$

These values illustrate some of the general properties presented in section 3; namely, that the NPI upper probability for successful system functioning given s successes in n tests is equal to the NPI lower probability for successful system functioning given $s+1$ successes in n tests. The value 0 (1) of the NPI lower (upper) probability for the case $s=0$ ($s=2$) reflects that in this case there is no strong evidence that the components can actually function (fail). These values emphasize the serious error that can be made if, instead of a careful analysis that combines the uncertainty modelling and available information, one would plug a 'reasonable' estimate of a parameter representing the functioning of a component into a formula for a probability for system functioning

depending on this parameter. For example, if it was assumed that a parameter θ represents an unknown probability for such a component to function, then, conditional on this parameter, the probability for successful functioning of this series system would be θ^{10}; for $s=1$, so one failure and one success in two tested components, a 'reasonable' estimate would be $\hat{\theta}=1/2$ (e.g. this would be both the moment estimate and maximum likelihood estimate), and it could be tempting to use this value to predict successful functioning of the series with probability $\hat{\theta}^{10}=(1/2)^{10}=1/1024$, which is far lower than the corresponding NPI lower probability 1/66 as presented in this paper; therefore, system reliability would be substantially underestimated. Although it is well known that to plug point estimates into such formulae is wrong, text books in reliability rarely address such issues carefully. In section 7, brief comments are made on the Bayesian approach, where it is easily seen that, for any reasonably non-informative prior distribution, the corresponding predictive probability of system functioning is within the interval created by the NPI-based lower and upper probabilities. The following is an informal argument that leads to a better alternative than the plug-in approach, and that is in line with the NPI approach. On the basis of one success in two tests, a predictive probability for the next component to be successful might logically be set at (about) 1/2. For the series system to function successfully, all ten components must function. Consider the second component in the series system, given that the first functions successfully, and given the data from the test; hence, for this component, the information available consists of two successful components out of three, and therefore a predictive probability for this component to function might be set at (about) 2/3. Continuing this reasoning, which acknowledges the interdependence of the ten components in the system, an intuitively reasonable predictive probability for successful functioning of the series system would be (about)

$$\frac{1}{2}\times\frac{2}{3}\times\frac{3}{4}\ldots\times\frac{10}{11}=\frac{1}{11}$$

which is between the corresponding NPI-based lower and upper probabilities. It should be emphasized that this latter informal reasoning is only presented as a possible explanation for why the use of a plug-in estimate is wrong; it is not suggested that the value 1/11 is a 'correct' precise probability in this case, an obvious reason being that this informal argument would lead to precise probability 0 (1) for system functioning in the case $s=0$ ($s=2$).

If, instead of a series system, a parallel system with ten components (so $k=1$, $m=10$) is considered, also with two components tested, then the NPI lower and upper probabilities for successful functioning of the system are

$$(\underline{P},\overline{P})(10:1|2,0)=\left(0,\frac{5}{6}\right)$$

$$(\underline{P},\overline{P})(10:1|2,1)=\left(\frac{5}{6},\frac{65}{66}\right)$$

$$(\underline{P},\overline{P})(10:1|2,2)=\left(\frac{65}{66},1\right)$$

Note here that mistakenly using a plug-in estimate of 1/2 for the case $s=1$, as discussed above, could lead to a suggestion that the value $1-(1/2)^{10}=1023/1024$ would be a reasonable probability for the event that the system will function. This value is substantially higher than the corresponding NPI upper probability 65/66, and far greater than the corresponding NPI lower probability 5/6, which might be attractive to use for risk assessment from a cautious perspective, and, hence, there could be a danger of overconfidence in the system's reliability if the uncertainty and information are not properly analysed.

Example 2

To illustrate further the NPI results for system reliability, presented in section 3, Table 1 provides the NPI lower and upper probabilities for all possible cases with $n=4$ components tested, of which s functioned successfully, and the system consisting of $m=5$ components, of which at least k must function.

The values in Table 1 show that, in order to obtain a reasonably large NPI lower probability for successful system functioning, it is not necessarily required that most tested components function well if k is small, which means that the system has much built-in redundancy, but for large values of k (nearly)

Table 1 NPI lower and upper probabilities for all cases with $m=5$ and $n=4$

	$k=1$		$k=2$		$k=3$		$k=4$		$k=5$	
	$\underline{P}$	$\overline{P}$	$\underline{P}$	$\overline{P}$	$\underline{P}$	$\overline{P}$	$\underline{P}$	$\overline{P}$	$\underline{P}$	$\overline{P}$
$s=0$	0	0.556	0	0.278	0	0.119	0	0.040	0	0.008
$s=1$	0.556	0.833	0.278	0.595	0.119	0.357	0.040	0.167	0.008	0.048
$s=2$	0.833	0.952	0.595	0.833	0.357	0.643	0.167	0.405	0.048	0.167
$s=3$	0.952	0.992	0.833	0.960	0.643	0.881	0.405	0.722	0.167	0.444
$s=4$	0.992	1	0.960	1	0.881	1	0.722	1	0.444	1

Table 2 NPI lower probabilities for zero-failure testing with $k=8$

	$m=8$	$m=9$	$m=10$	$m=11$	$m=12$
$s=n=5$	0.385	0.604	0.736	0.819	0.872
10	0.556	0.789	0.895	0.945	0.970
15	0.652	0.870	0.948	0.978	0.990

all tested components must have been successful. Table 1 shows that the lower and upper probabilities are (a) decreasing in k, keeping m, n, and s constant; (b) increasing in s, keeping m, n, and k constant.

Example 3

As mentioned in section 3, the results in this paper can also be used in zero-failure reliability demonstration from an NPI perspective, generalizing the results presented by Coolen and Coolen-Schrijner [**9**]. Suppose that for system functioning it is required that k components function, but that redundancy can be built into the system by increasing the total number of components m in the system. For example, components considered could be batteries required to provide back-up in case of problems with electricity supply for a safety critical system, where system functioning requires a minimum of three batteries to function when demanded, but where installing more batteries might provide important redundancy. Rahrouh *et al.* [**29**] presented a Bayesian approach for optimal decisions for reliability demonstration (see also reference [**30**]), assuming that only component tests with zero failures would lead to release of the system for practical use, as is often the case if high reliability is required. They considered both the costs of testing and the costs of extra system redundancy, and also took into account practical constraints with regard to test time and budget. Apart from cost and time figures, and related constraints, the key input for such decisions consists of the lower probabilities $\underline{P}(m:k|n,n)$, as functions of m and n for fixed k. To illustrate the approach presented in this paper, some such NPI lower probabilities are presented in Table 2, for $k=8$ and the cases $n=5$, 10, 15, and m varying from 8 to 12. Clearly, the corresponding NPI upper probabilities are all equal to one under the zero-failure testing assumption. Table 2 shows that the lower probability, under the assumption of zero-failure testing is (a) increasing in m, keeping n and k constant; (b) increasing in n, keeping m and k constant.

The NPI lower probabilities presented in Table 2 can be used in several ways. For example, consider the case $m=8$ with five zero-failure tests, leading to NPI lower probability 0.385 for successful system functioning. The table shows that increasing the redundancy to $m=9$, keeping $k=8$, would increase the NPI lower probability to 0.604, while to increase the number of zero-failure tests to ten would increase the NPI lower probability to 0.556; so if these two actions were available at similar costs, increase of redundancy might be preferred to more tests. However, if 15 tests were possible at a cost similar to the added redundancy, then this might be preferred, as the corresponding NPI lower probability would increase to 0.652, if all 15 tests would be successes. Clearly, extra testing has the added advantage of possibly finding more failures, in which case the analysis would be repeated after further inspection or development of the components. In the NPI approach, the absence of prior information makes it impossible to infer how likely failures in the tests would be, but in high reliability demonstration it would normally be considered to be quite surprising to encounter failures in tests.

Table 3 Required number n of zero-failure tests for $\underline{P}(m:8|n,n)\geqslant p$

	$m=8$	$m=9$	$m=10$	$m=11$	$m=12$
$p=0.75$	24	9	6	4	4
0.80	32	11	7	5	4
0.85	46	14	8	6	5
0.90	72	19	11	8	6
0.95	153	30	16	11	9
0.99	792	77	33	21	15

Table 3 extends this example by presenting the minimum number of zero-failure tests required to achieve a chosen value for the NPI lower probability for successful system functioning, again for $k=8$ and m varying from 8 to 12. The requirement considered is $\underline{P}(m:8|n,n)\geqslant p$ for different values of p.

The main conclusion from Table 3 is that, in order to demonstrate high reliability via zero-failure testing, many successful tests are required, yet the number can be reduced substantially by building in redundancy.

5 SERIES OF INDEPENDENT k_i-OUT-OF-m_i SUBSYSTEMS

The results of section 3 can be straightforwardly used to consider the reliability of systems that consist of a series configuration of $N\geqslant 2$ independent subsystems, with subsystem i ($i=1,\ldots,N$) a k_i-out-of-m_i system consisting of exchangeable components. Clearly, other system configurations could also be considered; these are left as an interesting topic for future research. As before, it is assumed that, in relation to subsystem i, n_i components that are exchangeable with those to be used in the subsystem have been tested, of which s_i functioned successfully.

For the series system to function, all of its subsystems must function, and due to the assumed independence of the subsystems (which implies independence of components in different subsystems), the NPI lower and upper probabilities for such a series system to function are

$$\underline{P}(\boldsymbol{m} : \boldsymbol{k}|\boldsymbol{n}, \boldsymbol{s}) = \prod_{i=1}^{N} \underline{P}(m_i : k_i|n_i, s_i) \tag{4}$$

and

$$\overline{P}(\boldsymbol{m} : \boldsymbol{k}|\boldsymbol{n}, \boldsymbol{s}) = \prod_{i=1}^{N} \overline{P}(m_i : k_i|n_i, s_i) \tag{5}$$

respectively, where notation with N-vectors $\boldsymbol{m}$, $\boldsymbol{k}$, $\boldsymbol{n}$, $\boldsymbol{s}$ has been introduced to generalize notation in an obvious manner from that used in section 3. Results for several special cases are easily derived from corresponding results in section 3; for example, the NPI lower probability for system functioning is equal to zero if $s_i = 0$ for at least one subsystem, and the NPI upper probability for system functioning is equal to one only if all components tested, for all subsystems, were successful, so $s_i = n_i$ for all $i = 1, \ldots, N$. In section 6, redundancy allocation for such series systems will be considered, under the assumption of zero-failure testing ($s_i = n_i$ for all $i = 1, \ldots, N$), in which case the NPI upper probability for the whole system to function is equal to one, and the corresponding NPI lower probability is equal to

$$\begin{aligned} \underline{P}(\boldsymbol{m} : \boldsymbol{k}|\boldsymbol{n}, \boldsymbol{n}) &= \prod_{i=1}^{N} \underline{P}(m_i : k_i|n_i, n_i) \\ &= \prod_{i=1}^{N} \left[1 - \binom{n_i + k_i - 1}{n_i} \binom{n_i + m_i}{n_i}^{-1} \right] \end{aligned} \tag{6}$$

Redundancy allocation is also of interest if some components failed the tests; this is left as an important and interesting topic for future research. The property (3) still holds per subsystem so that for the whole system

$$\overline{P}(\boldsymbol{m} : \boldsymbol{k}|\boldsymbol{n}, \boldsymbol{s}) = \underline{P}(\boldsymbol{m} : \boldsymbol{k}|\boldsymbol{n}, \boldsymbol{s} + \boldsymbol{1}) \tag{7}$$

where $\boldsymbol{1}$ is an N-vector with all components equal to 1. Using this property leads to a reduction in overall computational effort if it is wished to derive NPI lower and upper probabilities for system functioning for many or all possible test outcomes $\boldsymbol{s}$.

This generalization of the NPI approach to reliability assessment for a series of independent k_i-out-of-m_i subsystems is illustrated in Example 4.

Example 4

Consider a system that consists of two independent subsystems (so $N = 2$) in a series configuration, where, for each subsystem, three exchangeable components must function to ensure the subsystem to function, hence $k_1 = k_2 = 3$, and where five components exchangeable with those in subsystem 1 have been tested, and also five components exchangeable with those in subsystem 2 have been tested, so $n_1 = n_2 = 5$. Tables 4 and 5 present the NPI lower and upper probabilities, respectively, for functioning of this system, for varying numbers of test successes (s_1 and s_2) and different numbers of components (m_1 and m_2) in these k_i-out-of-m_i subsystems. Test results for which the NPI lower probability for system functioning is zero ($s_1 = 0$ or $s_2 = 0$) are deleted from Table 4; the case $s_1 = s_2 = 5$ is deleted from Table 5 as the corresponding NPI upper probability is one for all m_1 and m_2.

These tables illustrate the manner in which system reliability, measured by these NPI lower and upper probabilities, increases with increasing numbers of test successes, and with increasing system redundancy. They also illustrate property (7); for example, the reported NPI upper probabilities for $(s_1, s_2) = (3, 4)$ are equal to the corresponding NPI lower probabilities for $(s_1, s_2) = (4, 5)$. Note that in situations

Table 4 NPI lower probability for system functioning

$(m_1,m_2) =$		(3,4)	(4,4)	(3,5)	(3,6)	(4,5)	(5,5)
$(s_1,s_2) =$	(1,1)	0.0009	0.0023	0.0015	0.0022	0.0040	0.0069
	(1,2)	0.0030	0.0079	0.0047	0.0062	0.0125	0.0218
	(2,1)	0.0034	0.0079	0.0060	0.0087	0.0139	0.0218
	(2,2)	0.0119	0.0278	0.0187	0.0249	0.0437	0.0686
	(3,3)	0.0638	0.1276	0.0893	0.1086	0.1786	0.2500
	(4,3)	0.1276	0.2126	0.1786	0.2172	0.2976	0.3690
	(5,3)	0.2232	0.2976	0.3125	0.3801	0.4167	0.4583
	(3,4)	0.1063	0.2126	0.1318	0.1473	0.2636	0.3690
	(3,5)	0.1488	0.2976	0.1637	0.1705	0.3274	0.4583
	(4,4)	0.2126	0.3543	0.2636	0.2945	0.4393	0.5448
	(5,4)	0.3720	0.4960	0.4613	0.5154	0.6151	0.6766
	(4,5)	0.2976	0.4960	0.3274	0.3409	0.5456	0.6766
	(5,5)	0.5208	0.6944	0.5729	0.5966	0.7639	0.8403

Table 5 NPI upper probability for system functioning

$(m_1,m_2) =$		(3,4)	(4,4)	(3,5)	(3,6)	(4,5)	(5,5)
$(s_1,s_2) =$	(0,0)	0.0009	0.0023	0.0015	0.0022	0.0040	0.0069
	(1,0)	0.0034	0.0079	0.0060	0.0087	0.0139	0.0218
	(2,0)	0.0085	0.0170	0.0149	0.0216	0.0298	0.0417
	(0,1)	0.0030	0.0079	0.0047	0.0062	0.0125	0.0218
	(0,2)	0.0064	0.0170	0.0089	0.0109	0.0238	0.0417
	(1,1)	0.0119	0.0278	0.0187	0.0249	0.0437	0.0686
	(3,3)	0.2126	0.3543	0.2636	0.2945	0.4393	0.5448
	(4,3)	0.3720	0.4960	0.4613	0.5154	0.6151	0.6766
	(5,3)	0.5952	0.5952	0.7381	0.8247	0.7381	0.7381
	(3,4)	0.2976	0.4960	0.3274	0.3409	0.5456	0.6766
	(3,5)	0.3571	0.5952	0.3571	0.3571	0.5952	0.7381
	(4,4)	0.5208	0.6944	0.5729	0.5966	0.7639	0.8403
	(5,4)	0.8333	0.8333	0.9167	0.9545	0.9167	0.9167
	(4,5)	0.6250	0.8333	0.6250	0.6250	0.8333	0.9167

Table 6 NPI lower probability for system functioning after zero-failure testing

$(m_1,m_2)=$		(3,4)	(4,4)	(3,5)	(3,6)	(4,5)	(5,5)
$(n_1,n_2)=$	(2,2)	0.2400	0.3600	0.2857	0.3143	0.4286	0.5102
	(2,3)	0.2857	0.4286	0.3286	0.3524	0.4929	0.5867
	(2,5)	0.3333	0.5000	0.3667	0.3818	0.5500	0.6548
	(2,10)	0.3736	0.5604	0.3912	0.3967	0.5868	0.6986
	(3,2)	0.3000	0.4286	0.3571	0.3929	0.5102	0.5867
	(5,2)	0.3750	0.5000	0.4464	0.4911	0.5952	0.6548
	(10,2)	0.4615	0.5604	0.5495	0.6044	0.6672	0.6986
	(5,5)	0.5208	0.6944	0.5729	0.5966	0.7639	0.8403
	(10,5)	0.6410	0.7784	0.7051	0.7343	0.8562	0.8965
	(5,10)	0.5838	0.7784	0.6113	0.6198	0.8150	0.8965
	(10,10)	0.7185	0.8725	0.7523	0.7629	0.9135	0.9565
	(50,50)	0.9394	0.9916	0.9430	0.9434	0.9954	0.9992
	(50,51)	0.9396	0.9918	0.9431	0.9434	0.9954	0.9993
	(50,55)	0.9401	0.9923	0.9431	0.9434	0.9955	0.9993

where for a particular subsystem all performed tests are successes, the NPI upper probability for system functioning is in fact the NPI upper probability that the other subsystem functions. For example, in Table 5, for $(s_1, s_2) = (5, 4)$, the NPI upper probabilities for system functioning when $(m_1, m_2) = (3, 5)$, (4, 5), and (5, 5) are identical and equal to the NPI upper probability that subsystem 2, a 3-out-of-5 subsystem, functions.

For the same system [$k_1 = k_2 = 3$, with varying (m_1, m_2)], Table 6 presents the NPI lower probabilities for system functioning after zero-failure testing, so with $(s_1, s_2) = (n_1, n_2)$, for different numbers of tests performed. Comparing, for example, the lower probabilities for $(n_1, n_2) = (10, 5)$ with those for $(n_1, n_2) = (5, 10)$, illustrates the intuitively logical property that it is better to have more zero-failure tests for subsystems with less redundancy than for subsystems with more redundancy, if both subsystems have equal k_i. Also, the increase in reliability due to extra zero-failure tests clearly decreases as a function of the number of such tests already performed.

6 REDUNDANCY ALLOCATION AFTER ZERO-FAILURE TESTING

In this section, the systems considered are again systems consisting of a series configuration of independent k_i-out-of-m_i subsystems, with the restriction to zero-failure testing for all components, so n_i components that are exchangeable to those in subsystem i have been tested, and none failed, hence $s_i = n_i$ for all $i = 1, \ldots, N$. For this situation, the problem considered here is how to assign extra components to subsystems, in order to increase the redundancy, and hence the overall system reliability, which is assumed to be quantified by the NPI lower probability for system functioning (the corresponding NPI upper probability is one). It is particularly logical to focus attention on the NPI lower probability in situations where a specific reliability requirement must be met, as the lower probability can be considered to be a conservative inference. It should be remarked that such redundancy allocation is, of course, also interesting to study, related to general test results; this is left as an important topic for future research.

In traditional reliability theory, redundancy allocation for systems tends to involve complex computations in order to achieve optimality [**31**]. In the NPI setting for the systems considered here with zero-failure testing, however, optimal redundancy allocation is achieved in a straightforward manner if only overall system reliability is considered (so, for example, no costs of components are taken into account). It is proven here that sequential one-step optimal allocation of extra components always leads to optimal allocation of any number of extra components. The algorithm is described next, and extensively illustrated and discussed in Example 5. The claim of optimality of this algorithm is justified in Appendix B.

The NPI lower probability for successful functioning of the whole system, following n_i zero-failure tests for components exchangeable with those in subsystem i, was given as equation (6), which was

$$\underline{P}(\boldsymbol{m} : \boldsymbol{k} | \boldsymbol{n}, \boldsymbol{n}) = \prod_{i=1}^{N} \underline{P}(m_i : k_i | n_i, n_i) = \prod_{i=1}^{N} \left[1 - \binom{n_i + k_i - 1}{n_i} \binom{n_i + m_i}{n_i}^{-1} \right]$$

Suppose now that j_i additional components are added to subsystem i, for $i = 1, \ldots, N$, with no further tests performed, then the NPI lower probability for successful functioning of the system becomes (as before with obvious vector notation)

$$\underline{P}(\boldsymbol{m} + \boldsymbol{j} : \boldsymbol{k} | \boldsymbol{n}, \boldsymbol{n}) = \prod_{i=1}^{N} \underline{P}(m_i + j_i : k_i | n_i, n_i) = \prod_{i=1}^{N} \left[1 - \binom{n_i + k_i - 1}{n_i} \times \binom{n_i + m_i + j_i}{n_i}^{-1} \right] \quad (8)$$

Two situations of most practical interest are the optimal allocation of components to subsystems when the opportunity arises to add a given number of further components, and the requirement to add the minimum number of components needed to satisfy a reliability criterion that the NPI lower probability of system functioning has to exceed a set value. Both of these problems, and all other related problems where it may be wished to add extra components in an optimal sequence, are solved by the following

algorithm (given below in pseudo-code), in which, for $i=1,\ldots,N$ and $j_i \geqslant 0$,

$$\rho(i,j_i) = \frac{\underline{P}(m_i + j_i + 1 : k_i | n_i, n_i)}{\underline{P}(m_i + j_i : k_i | n_i, n_i)}$$

Hence, $\rho(i,j_i)$ is the factor with which the NPI lower probability of successful functioning of subsystem i increases when j_i+1 instead of j_i extra components are added to subsystem i. Clearly, $\rho(i,j_i) > 1$ for all i, j_i.

Optimal allocation algorithm

1. Set $j_i = 0$ and calculate $\rho(i,j_i) = \rho(i,0)$ for all $i=1,\ldots,N$;
2. Determine i_m such that
   $$\rho(i_m, j_{i_m}) = \max_{1 \leqslant i \leqslant N} \rho(i,j_i)$$
   If this i_m is not a unique value, then pick any one of these values;
3. Add an extra component to subsystem i_m: set $j_{i_m} := j_{i_m} + 1$ and calculate $\rho(i_m, j_{i_m})$;
4. Return to step 2, using the same values $\rho(i,j_i)$ as in the previous step for $i \neq i_m$, together with the new value $\rho(i_m, j_{i_m})$ for subsystem i_m, as just calculated in steps 2 and 3.

This algorithm can be stopped at any time, normally if a stop-criterion is achieved, e.g. if the total number of possible extra components has been assigned, or if a predetermined value for the NPI lower probability for system functioning has been achieved. After stopping the algorithm, the vector $\boldsymbol{j} = (j_1,\ldots,j_N)$ gives the number of extra components that is added to each subsystem and the NPI lower probability for successful functioning of the system after adding these extra components is equal to

$$\underline{P}(\boldsymbol{m}+\boldsymbol{j} : \boldsymbol{k} | \boldsymbol{n}, \boldsymbol{n}) = \underline{P}(\boldsymbol{m} : \boldsymbol{k} | \boldsymbol{n}, \boldsymbol{n}) \times \prod_{i=1}^{N} \prod_{l_i=0}^{j_i-1} \rho(i, l_i) \quad (9)$$

This enables easy calculation of the NPI lower probability at step 3 in the above algorithm, as it just requires the previous value of this NPI lower probability to be multiplied by the $\rho(i_m, j_{i_m})$ calculated at that step.

The claim that this allocation algorithm is optimal is justified in Appendix B, the key to the proof being the fact that $\rho(i,j_i)$ is strictly decreasing in j_i for all $i=1,\ldots,N$. Variations to this algorithm are obvious for situations with some restrictions on the allocation opportunities. The situation becomes more complicated if costs of additional components, and possibly overall budgets, must be taken into account; some brief comments are included in section 7, but general NPI theory for optimal redundancy allocation under cost considerations, and possibly also taking costs of testing into account, is left as an interesting and important topic for future research. Example 5 illustrates the optimal allocation algorithm, and discusses many relevant aspects of this approach.

Example 5

Throughout this example, the starting point is a system consisting of four independent k_i-out-of-m_i subsystems in series configuration, with the values k_i and m_i as given in Table 7. Several scenarios of allocation of additional components, to increase redundancy optimally, will be illustrated for this system, all assuming n_i zero-failure tests for subsystem i.

In the following tables, optimal allocation sequences are presented using notation i_t and w_t, which is such that the t-th additional component is added to subsystem i_t, and it is the w_tth extra component allocated to that subsystem. The NPI lower probability for successful functioning of subsystem i_t, which is here considered to represent the reliability of this subsystem, after the t-th extra component has been added to this subsystem, is denoted by $R(i_t)$, and $RS(t)$ denotes the NPI lower probability for successful functioning of the whole system at that stage. Three cases are considered, in each of which five extra components are optimally allocated, namely with $n_i = 5$, 10, 50 zero-failure tests for each subsystem. The optimal allocation sequences are presented in Tables 8, 10, and 12, and the corresponding initial and final system reliability (NPI lower probabilities),

Table 7 Subsystem i: k_i-out-of-m_i

i	k_i	m_i
1	1	2
2	2	3
3	3	5
4	1	4

Table 8 Optimal allocation sequence of five components – $n_i = 5$

t	i_t	w_t	$R(i_t)$	$RS(t)$
1	2	1	0.9524	0.8248
2	3	1	0.9545	0.8589
3	1	1	0.9821	0.8858
4	2	2	0.9762	0.9079
5	3	2	0.9735	0.9259

Table 9 Reliability before and after optimal redundancy allocation of five components – $n_i = 5$

(Sub)system	Initial reliability	Final reliability
1	0.9524	0.9821
2	0.8929	0.9762
3	0.9167	0.9735
4	0.9921	0.9921
RS	0.7733	0.9259

for each subsystem and for the whole system (*RS*), are presented in Tables 9, 11, and 13.

Note that, for $n_i = 10$, the five extra components are added to subsystems in the same sequence as in the case with $n_i = 5$, but the order of components added is different for $n_i = 50$. With two components added to subsystem 1, it has become a 1-out-of-4 system, just like subsystem 4 (with no components added), leading to the same final reliability values for these two subsystems in Table 13.

With the help of the system considered in this example (Table 7) it is illustrated, in more detail, how the optimal allocation sequence of additional components to subsystems can change with the number of zero-failure tests performed per subsystem. The number of zero-failure tests for subsystems 1, 3, and 4 are kept at 5 (so $n_i = 5$ for $i = 1, 3, 4$), but for subsystem 2 the number of zero-failure tests, n_2, is varied from 5 to 10. The optimal sequences in which five further components are allocated to the subsystems are presented in Table 14, together with the final reliability for the whole system. This clearly illustrates the intuitively logical effect that, with an increasing number of zero-failure tests for components exchangeable with those in subsystem 2, the benefit of additional redundancy added to subsystem 2 decreases, and, hence, the optimal redundancy allocation order changes with subsystem 2 moving back in the order.

To finish this example, the optimal allocation sequence of 12 further components to subsystems is presented in Table 15, for the same system as considered throughout this example (Table 7) and with $n_i = 5$ for all $i = 1, \ldots, 4$ (so the first five allocations are identical to those presented in Table 8). Table 16

Table 10 Optimal allocation sequence of five components – $n_i = 10$

t	i_t	w_t	$R(i_t)$	$RS(t)$
1	2	1	0.9890	0.9517
2	3	1	0.9918	0.9650
3	1	1	0.9965	0.9765
4	2	2	0.9963	0.9837
5	3	2	0.9966	0.9885

Table 11 Reliability before and after optimal redundancy allocation of five components – $n_i = 10$

(Sub)system	Initial reliability	Final reliability
1	0.9848	0.9965
2	0.9615	0.9963
3	0.9780	0.9966
4	0.9990	0.9990
RS	0.9252	0.9885

Table 12 Optimal allocation sequence of five components – $n_i = 50$

t	i_t	w_t	$R(i_t)$	$RS(t)$
1	2	1	0.9998387	0.9987007
2	1	1	0.9999573	0.9994118
3	3	1	0.9999592	0.9997521
4	2	2	0.9999853	0.9998987
5	1	2	0.9999968	0.9999382

Table 13 Reliability before and after optimal redundancy allocation of five components – $n_i = 50$

(Sub)system	Initial reliability	Final reliability
1	0.9992459	0.9999968
2	0.9978229	0.9999853
3	0.9996188	0.9999592
4	0.9999968	0.9999968
RS	0.9966872	0.9999382

Table 14 Allocation of extra components, with n_2 varying – $n_i = 5$ for $i = 1, 3, 4$

n_2	allocation order	final reliability
5	2, 3, 1, 2, 3	0.9259
6	2, 3, 1, 3, 2	0.9341
7	2, 3, 1, 3, 2	0.9389
8	3, 2, 1, 3, 2	0.9419
9	3, 2, 1, 3, 3	0.9450
10	3, 1, 2, 3, 3	0.9479

Table 15 Optimal allocation sequence of 12 components – $n_i = 5$

t	i_t	w_t	$R(i_t)$	$RS(t)$	t	i_t	w_t	$R(i_t)$	$RS(t)$
1	2	1	0.9524	0.8248	7	3	3	0.9837	0.9460
2	3	1	0.9545	0.8589	8	1	2	0.9921	0.9556
3	1	1	0.9821	0.8858	9	3	4	0.9895	0.9612
4	2	2	0.9762	0.9079	10	2	4	0.9924	0.9665
5	3	2	0.9735	0.9259	11	1	3	0.9960	0.9704
6	2	3	0.9870	0.9362	12	4	1	0.9960	0.9742

Table 16 Reliability before and after optimal redundancy allocation of 12 components – $n_i = 5$

(Sub)system	Initial reliability	Final reliability
1	0.9524	0.9960
2	0.8929	0.9924
3	0.9167	0.9895
4	0.9921	0.9960
RS	0.7733	0.9742

presents the corresponding initial and final reliability values per subsystem and for the whole system (*RS*).

The optimal allocation sequence in Table 15 is not unique, as the allocations at steps 11 and 12 could have been performed in a different order, leading to the same result, as these two allocations have identical effects on the NPI lower probability for the functioning of the whole system. This follows immediately from the fact that, after step 8, subsystem 1 has become a 1-out-of-4 system, just like subsystem 4. If such redundancy allocation had been performed in order to achieve a system reliability, expressed through the NPI lower probability of successful system functioning, of at least 0.95, then Table 15 shows that a total of eight extra components would be sufficient, with two extra components allocated to subsystem 1, and three each to subsystems 2 and 3. It is also clearly illustrated that the effect of such extra components on the overal system reliability decreases as a function of the total number of already added components.

7 CONCLUDING REMARKS

The Bayesian approach to statistics also provides a natural framework for inferences of the kind considered in this paper. If a parameter θ is assumed to represent the probability of successful functioning of a single component, then the probability of successful functioning of a k-out-of-m system, as a function of θ, is simply represented by

$$P(m : k|\theta) = \sum_{j=k}^{m} \binom{m}{j} \theta^j (1-\theta)^{m-j}$$

For any assumed prior probability distribution for θ and test data, the Bayesian approach leads to a precise posterior probability for successful functioning of the system. Beta distributions are particularly attractive prior distributions for θ in this case, as they are conjugate, which means that the corresponding posterior distributions for θ are also beta distributions. The results presented in this paper actually coincide with the corresponding Bayesian results based on two particular beta prior distributions, namely the NPI lower probabilities for successful system functioning correspond to Bayesian probabilities based on the beta(0,1) prior, and the NPI upper probabilities correspond similarly to the beta (1,0) prior (note that these priors are improper, but the corresponding posterior predictive probabilities of interest here do exist). This is due to the fact that, generally, for events of the form 'k or more successes out of m trials', the inferences of Coolen [**18**] coincide with these Bayesian inferences. It should, however, be emphasized that this is not the case for all events considered in the NPI approach by Coolen [**18**]. The fact that these inferences provide the same values for the (lower and upper) probabilities considered can be understood from the representation of successes and failures that underlies NPI [**18**], and which is closely related to the approach by Bayes [**19**]. This underlying representation, which is not discussed further here, also provides a simple justification for the equality (3) presented in section 3. In relation to Example 1, it is useful to remark that, when using a Bayesian approach with improper prior beta(0,0), and adding test data consisting of one success and one failure (leading to a uniform posterior distribution for θ), the posterior probability for successful functioning of a ten-out-of-ten system would be equal to 1/11, the value also derived via an informal argument in Example 1.

Hartigan [**32**] proposed the use of either the beta (0,1) or the beta (1,0) prior for 'cautious' inference, and Coolen and Coolen-Schrijner [**33**] also proposed the beta(0,1) prior for Bayesian high-reliability demonstration, mainly due to its relation to NPI. However, from Bayesian perspective, there is no strongly convincing argument for such a particular choice of prior distribution, yet the precise choice of prior distribution always influences the inferences to some extent. Even more, the NPI approach is based on fewer assumptions than the Bayesian approach, as only $A_{(n)}$, which is related to finite exchangeability, is assumed, whereas the assumption of an unknown parameter θ to represent the probability of successful functioning for each component requires an underlying assumption of an infinite population of such components, all of which are assumed to be exchangeable [**34**]. The impact of this latter assumption is often not clear, and might appear to be of little relevance as the NPI results in this paper agree with some Bayesian results, as just mentioned. However, it should be emphasized that the finite exchangeability assumption that underlies the NPI approach does not lead to the assumed existence of a single parameter θ representing the probability of successful functioning of each component considered. It could be argued, therefore, that the Bayesian approach uses a detour, via a stronger exchangeability assumption on an infinite population of components, to, in this case, obtain to a similar answer as provided by the more direct NPI approach, with different Bayesian prior distributions corresponding to the NPI lower and upper probabilities.

It is an advantage of NPI that the inferences are in terms of lower and upper probabilities, as these naturally reflect the amount of information available, and deal in an attractive manner with situations where all test results are failures or all are successes.

In practical risk assessment, it is often clear which of the lower and upper probabilities should be used to support decisions, with lower probability for system functioning often the more natural in reliability applications, in particular if the aim is to demonstrate reliability [**30**], while the difference between corresponding upper and lower probabilities can provide further useful information.

In sections 5 and 6, more complex systems were considered than in section 3 of this paper, but still with an assumption of independence between different subsystems that enabled quite straightforward application of the NPI approach. More challenging is the development of the NPI approach for systems consisting of parallel and series subsystems, and beyond that for more general system structures; in particular, if exchangeable components are used in different subsystems, as, for such systems, the basic NPI results by Coolen [**18**] must be extended to take the particular groupings of future components in the system into account. The basic idea of the NPI approach [**18**] will remain the same, but the combinatorics involved in deriving the lower and upper probabilities will be challenging for larger systems, which raises interesting problems for future research.

The results in section 6 determine how to allocate additional components optimally for redundancy for any criterion in the case where the cost of components is irrelevant, or where these are the same for all components. If the costs of additional components differ per subsystem, say c_i for each additional component for subsystem i, and the aim is to maximize system reliability under budget constraints, then the redundancy allocation problem becomes more complex. As the maximization of $\prod_{i=1}^{N} \prod_{l_i=0}^{j_i-1} \rho(i, l_i)$ can be replaced by maximization of $\sum_{i=1}^{N} \sum_{l_i=0}^{j_i-1} \ln[\rho(i, l_i)]$, this problem is close in nature to the well-known knapsack problems in discrete optimization [**32**, **36**]. If the $\rho(i, j_i)$ were constant, then indeed the redundancy allocation problem with different costs for components and with a budget constraint would be a knapsack problem, and could therefore be solved by available algorithms. However, the fact that, for each i, $\rho(i, j_i)$ is decreasing in j_i, complicates matters, and leaves an interesting topic for future research. It is expected that knapsack solution methods [**35**, **36**] can be modified in order to solve this problem, and they might also provide suitable approximate solutions if large systems with many additional components are considered.

Throughout this paper, several research challenges related to this NPI approach to system reliability and redundancy allocation have been mentioned. Several of these require some further development of NPI theory and methods, whereas others provide further analytical challenges or require development of suitable computational algorithms. It will also be interesting to compare the NPI approach for system reliability, presented in this paper, in detail with alternative approaches, if possible in actual applications.

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APPENDIX A

Notation

$A_{(n)}$	Hill's inferential assumption
c_i	costs to add one extra component to subsystem i
i_t	the subsystem to which the t-th extra component is added
$\boldsymbol{j}$	vector $(j_1, \ldots, j_N)$
J	total number of additional components added to the whole system ($J = \sum_{i=1}^{N} j_i$)
j_i	number of additional components added to subsystem i
k	minimum number of functioning components for a k-out-of-m system to function
$\boldsymbol{k}$	vector $(k_1, \ldots, k_N)$
k_i	minimum number of functioning components for a k_i-out-of-m_i subsystem to function
m	total number of components in a k-out-of-m system
$\boldsymbol{m}$	vector $(m_1, \ldots, m_N)$
m_i	total number of components in a k_i-out-of-m_i subsystem
n	number of components that have been tested in a k-out-of-m system
$\boldsymbol{n}$	vector $(n_1, \ldots, n_N)$
N	number of independent subsystems in the system
n_i	number of components that have been tested in a k_i-out-of-m_i subsystem
p	chosen value of the NPI lower probability for successful system functioning
$\underline{P}(A)$	NPI lower probability for event A
$\overline{P}(A)$	NPI upper probability for event A
R_{i_t}	NPI lower probability for successful functioning of subsystem i_t after the tth extra component has been added to this subsystem
R_t	set $\{r_1, \ldots, r_t\}$ containing the numbers of successes out of m future trials in the event of interest, with $0 \leqslant r_1 < \ldots < r_t \leqslant m$ and $1 \leqslant t \leqslant m+1$
$RS(t)$	NPI lower probability for successful functioning of the whole system after t extra components are added to the system
s	number of successfully tested components (out of n) for a k-out-of-m system
$\boldsymbol{s}$	vector $(s_1, \ldots, s_N)$
s_i	number of successfully tested components (out of n_i) for a k_i-out-of-m_i subsystem
w_t	number of extra components (out of t) that have been added to subsystem i_t
Y_a^b	random number of successes in trials a to b

θ unknown parameter
$\rho(i, j_i)$ factor with which the NPI lower probability of successful functioning of subsystem i increases when $j_i + 1$ instead of j_i additional components are added

APPENDIX B

Optimality of redundancy allocation algorithm

In section 6, it was claimed that the presented redundancy allocation algorithm, in which sequentially each additional component is allocated according to one-step optimality, provides overall optimality in the sense of maximum NPI lower probability for successful functioning of the system, no matter how many components can be added in total, or indeed how the number of extra components is determined. Equation (9) implies that, for any total number $J \geqslant 1$ of components that can be added to the subsystems, optimality is achieved by maximization of $\prod_{i=1}^{N} \prod_{l_i=0}^{j_i-1} \rho(i, l_i)$ under the constraint $\sum_{i=1}^{N} j_i \leqslant J$. As $\rho(i, j_i) > 1$ for all i, j_i, this maximum will be attained with the active constraint $\sum_{i=1}^{N} j_i = J$, so all additional components available or allowed are used. The proof that this product $\prod_{i=1}^{N} \prod_{l_i=0}^{j_i-1} \rho(i, l_i)$ is maximized, under the active constraint, by the sequence of one-step optimal allocations, follows from the fact that, for each $i = 1, \ldots, N$, $\rho(i, j_i)$ is strictly decreasing in j_i. The proof of this property is given below.

Consider a specific subsystem i, and for simplicity denote the j_i by j, and let

$$a = \binom{n_i + k_i - 1}{n_i}, \quad A_j = \binom{n_i + m_i + j}{n_i}$$

Observe that $A_{j+1} = [1 + n_i/(m_i + j + 1)]A_j$, for $j \geqslant 0$, and write

$$\frac{1}{\rho(i,j)} = \frac{1 - a/A_j}{1 - a/A_{j+1}} = \frac{A_{j+1} - a - an_i/(m_i + j + 1)}{A_{j+1} - a} = 1 - \frac{an_i}{(m_i + j + 1)(A_{j+1} - a)}$$

As $A_0 > a$, and the sequences $\{A_{j+1} - a\}_j$ and $\{m_i + j + 1\}_j$ are strictly increasing in j, also $1/\rho(i, j)$ is strictly increasing in j. Hence, the sequence $\{\rho(i, j)\}_{j \geqslant 1}$ is strictly decreasing.