SOLUTIONS TO THE PROBLEMS

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Problem 1

First of all, it follows from

$$f = f(a)I_{\{a\}} + f(b)I_{\{b\}}$$

and the linearity of a precise prevision P that

$$P(f) = f(a)P(\{a\}) + f(b)P(\{b\}).$$

If we now let $\alpha = P(\{a\})$ then by linearity it follows that α belongs to $[0,1]$ and that $P(\{b\}) = 1 - P(\{a\}) = 1 - \alpha$. This proves the first point.

The second point is trivial. Just calculate the values of the two alternative expressions in a and b. They are both $f(a)$ and $f(b)$ respectively.

To prove the third point, consider any coherent lower prevision $\underline{P}$ and any gamble f. Assume first that $f(a) \geq f(b)$. Then the left hand side is given by

$$\varepsilon\,[\alpha f(a) + (1-\alpha)f(b)] + (1-\varepsilon)f(b) = \varepsilon\alpha f(a) + (1-\varepsilon\alpha)f(b),$$

and for the right hand side, use coherence and the second point to see that it is equal to

$$f(b) + [f(a) - f(b)]\underline{P}(\{a\}) = f(b) + [f(a) - f(b)]\varepsilon\alpha = \varepsilon\alpha f(a) + (1-\varepsilon\alpha)f(b)$$

as well. Assume next that $f(b) \geq f(a)$. Then the left hand side is given by

$$\varepsilon\,[\alpha f(a) + (1-\alpha)f(b)] + (1-\varepsilon)f(a) = \varepsilon(1-\alpha)f(b) + [1-\varepsilon(1-\alpha)]f(a),$$

and for the right hand side, use coherence and the second point to see that it is equal to

$$\begin{aligned} f(a) + [f(b) - f(a)]\underline{P}(\{b\}) &= f(a) + [f(b) - f(a)]\varepsilon(1-\alpha) \\ &= \varepsilon(1-\alpha)f(b) + [1-\varepsilon(1-\alpha)]f(a). \end{aligned}$$

Problem 2

Let $\underline{P}_1$ be the uniform (precise) prevision $P_{\frac{1}{2}}$ on $\mathscr{L}(\mathscr{X}_1)$, and let $\underline{P}_2$ be the vacuous lower prevision min on $\mathscr{L}(\mathscr{X}_2)$.

Then for any gamble f on $\mathscr{X}_1 \times \mathscr{X}_2$:

$$\underline{P}_1(\underline{P}_2(f)) = \frac{1}{2}\min_{x_2\in\mathscr{X}} f(a,x_2) + \frac{1}{2}\min_{x_2\in\mathscr{X}} f(b,x_2) = \frac{1}{2}\left[\min_{x_2\in\mathscr{X}} f(a,x_2) + \min_{x_2\in\mathscr{X}} f(b,x_2)\right],$$

whereas

$$\underline{P}_2(\underline{P}_1(f)) = \min_{x_2\in\mathscr{X}}\left[\frac{1}{2}f(a,x_2) + \frac{1}{2}f(b,x_2)\right] = \frac{1}{2}\min_{x_2\in\mathscr{X}}\left[f(a,x_2) + f(b,x_2)\right].$$

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PROBLEM 3

Observe that

$$\begin{aligned}\underline{P}_1(\underline{P}_2(f)) &= \sum_{k=1}^{n} m_1(F_k) \min_{x_1\in F_k} \underline{P}_2(f(x_1,\cdot)) \\ &= \sum_{k=1}^{n} m_1(F_k) \min_{x_1\in F_k}\left[\sum_{\ell=1}^{n} m_2(G_\ell) \min_{x_2\in G_\ell} f(x_1,x_2)\right] \\ &\geq \sum_{k=1}^{n} m_1(F_k) \sum_{\ell=1}^{n} m_2(G_\ell) \min_{x_1\in F_k}\min_{x_2\in G_\ell} f(x_1,x_2) \\ &= \underline{P}_1 \times_D \underline{P}_2(f).\end{aligned}$$

PROBLEM 4

For the first point, $\pi^t f(h) = f(\pi(h)) = f(t) = 2$ and $\pi^t f(t) = f(\pi(t)) = f(h) = -1$. For the second point,

$$\pi P(f) = P(\pi^t f) = \frac{1}{2}\pi^t f(h) + \frac{1}{2}\pi^t f(t) = \frac{2-1}{2} = \frac{1}{2}.$$

Observe that

$$P(f) = \frac{1}{2} f(h) + \frac{1}{2} f(t) = \frac{-1+2}{2} = \frac{1}{2}.$$

PROBLEM 5

Taking into account the solution to Problem 1, we have to look for values of α and ε such that both

$$\begin{aligned}\varepsilon x(2\alpha-1) - (1-\varepsilon)x = 0 \\ \varepsilon x(2\alpha-1) + (1-\varepsilon)x = 0\end{aligned}$$

for all real x, whence $\varepsilon = 1$ and $\alpha = \frac{1}{2}$.

PROBLEM 6

For the first point, there are two permutations, characterised by

$$\begin{pmatrix} a & a \\ b & b \end{pmatrix} \text{ and } \begin{pmatrix} a & b \\ b & a \end{pmatrix}.$$

The only invariant atom is clearly $\{a,b\}$.

For the third point, use the characterisation in Problem 1 and weak invariance to find

$$\alpha\varepsilon = \underline{P}(\{a\}) = \underline{P}(\{b\}) = (1-\alpha)\varepsilon.$$

This means that either $\varepsilon = 0$ or $\alpha = \frac{1}{2}$, which proves the third point.

By the Fundamental Theorem for Strong Permutation invariance, the strongly permutation invariant lower previsions are given by

$$P^u(f|\{a,b\}) = \frac{1}{2}[f(a)+f(b)] = P_{\frac{1}{2}}(f),$$

and it is actually a unique linear prevision.

PROBLEM 7

Observe that, with $A_1 = \{1,3,5\}$ and $A_2 = \{2,4,6\}$, we have that

$$P^u(f|A_1) = \frac{f(1)+f(3)+f(5)}{3} \text{ and } P^u(f|A_2) = \frac{f(2)+f(4)+f(6)}{3}.$$

By the Fundamental Theorem for Strong Permutation invariance, the strongly permutation invariant lower previsions are given by

$$\underline{P}(f) = \underline{Q}(P^u(f|\cdot))$$

where $\underline{Q}$ is any coherent lower prevision on the two-element space $\{A_1, A_2\}$, and therefore, by Problem 1, of the form

$$\underline{Q}(g) = \varepsilon\,[\alpha g(A_1) + (1-\alpha)g(A_2)] + (1-\varepsilon)\min\{g(A_1), g(A_2)\}.$$

Now let $g = P^u(f|\cdot)$ to find

$$\underline{P}(f) = \varepsilon\left[\alpha\frac{f(1)+f(3)+f(5)}{3} + (1-\alpha)\frac{f(2)+f(4)+f(6)}{3}\right] + (1-\varepsilon)\min\left\{\frac{f(1)+f(3)+f(5)}{3}, \frac{f(2)+f(4)+f(6)}{3}\right\},$$

where α and ε can take any value in $[0,1]$. This solves the first point. For the second point, observe that the linearity requires that $\varepsilon = 1$.